On the randomized complexity of Banach space valued integration

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Abstract

We study the complexity of Banach space valued integration in the randomized setting. We are concerned with r-times continuously differentiable functions on the d-dimensional unit cube Q, with values in a Banach space X, and investigate the relation of the optimal convergence rate to the geometry of X. It turns out that the n-th minimal errors are bounded by $cn^{-r/d-1+1/p}$ if and only if X is of equal norm type p.

1 Introduction

Integration of scalar valued functions is an intensively studied topic in the theory of information-based complexity, see [12], [10], [11]. Motivated by applications to parametric integration, recently the complexity of Banach space valued integration was considered in [2]. It was shown that the behaviour of the n-th minimal errors e_n^{ran} of randomized integration in $C^r(Q, X)$ is related to the geometry of the Banach space X in the following way: The infimum of the exponents of the rate is determined by the supremum of p such that X is of type p. In the present paper we further investigate this relation. We establish a connection between n-th minimal errors and equal norm type p constants for n vectors. It follows that e_n^{ran} is bounded by $cn^{-r/d-1+1/p}$ if and only if X is of equal norm type p.

2 Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We introduce some notation and concepts from Banach space theory needed in the sequel. For Banach spaces X and Y let B_X be the closed unit ball of X and $\mathscr{L}(X, Y)$ the space of bounded linear operators from X to Y, endowed with the usual norm. If $X = Y$, we write $\mathscr{L}(X)$. The norm of X is denoted by $\|\cdot\|$, while other norms are distinguished by subscripts. We assume that all considered Banach spaces are defined over the same scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Let $Q = [0, 1]^d$ and let $C^r(Q, X)$ be the space of all r-times continuously differentiable functions $f : Q \to X$ equipped with the norm

$$\|f\|_{C^r(Q,X)} = \max_{0\le|\alpha|\le r,\, t\in Q} \|D^\alpha f(t)\|,$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$, $|\alpha| = |\alpha_1| + \dots + |\alpha_d|$ and D^α denotes the respective partial derivative. For $r = 0$ we write $C^0(Q, X) = C(Q, X)$, which is the space of continuous X-valued functions on Q. If $X = \mathbb{K}$, we write $C^r(Q)$ and $C(Q)$.

Let $1 \le p \le 2$. A Banach space X is said to be of (Rademacher) type p, if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$

$$\left(\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|^p\right)^{1/p} \le c\left(\sum_{k=1}^n \|x_i\|^p\right)^{1/p}, \tag{1}$$

where $(\varepsilon_i)_{i=1}^n$ is a sequence of independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_i = -1\} = \mathbb{P}\{\varepsilon_i = +1\} = 1/2$ on some probability space $(\Omega, \Sigma, \mathbb{P})$ (we refer to [9, 7] for this notion and related facts). The smallest constant satisfying (1) is called the type p constant of X and is denoted by $\tau_p(X)$. If there is no such $c > 0$, we put $\tau_p(X) = \infty$. The space $L_{p_1}(\mathcal{N}, \nu)$ with $(\mathcal{N}, \nu)$ an arbitrary measure space and $p_1 < \infty$ is of type p with $p = \min(p_1, 2)$.

Furthermore, given $n \in \mathbb{N}$, let $\sigma_{p,n}(X)$ be the smallest $c > 0$ for which (1) holds for any $x_1, \dots, x_n \in X$ with $\|x_1\| = \dots = \|x_n\|$. The contraction principle for Rademacher series, see ([7], Th. 4.4), implies that $\sigma_{p,n}(X)$ is the smallest constant $c > 0$ such that for $x_1, \dots, x_n \in X$

$$\left(\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|^p\right)^{1/p} \le c n^{1/p} \max_{1\le i\le n} \|x_i\|. \tag{2}$$

We say that X is of equal norm type p, if there is a constant $c > 0$ such that $\sigma_{p,n}(X) \le c$ for all $n \in \mathbb{N}$. Clearly, $\sigma_{p,n}(X) \le \tau_p(X)$ and type p implies equal norm type p.

Let us comment a little more on the relation of the different notions of type which are used here and in the literature. The concept of equal norm type p was first introduced and used by R. C. James in the case $p = 2$ in [6]. There it is shown that X is of equal norm type 2 if and only if X is of type 2. This result is attributed to G. Pisier. Later, it even turned out in [1] that the sequence $\sigma_{2,n}(X)$ and the corresponding sequence $\tau_{2,n}(X)$ of type 2 constants computed with n vectors are uniformly equivalent. In contrast, for $1 < p < 2$, L. Tzafriri [13] constructed Tsirelson spaces without type p but with equal norm type p. Finally, V. Mascioni introduced and studied the notion of weak type p for $1 < p < 2$ in [8] and showed that, again in contrast to the situation for $p = 2$, a Banach space X is of weak type p if and only if it is of equal norm type p.

Throughout the paper $c, c_1, c_2, \dots$ are constants, which depend only on the problem parameters r, d, but depend neither on the algorithm parameters n, l etc. nor on the input f. The same symbol may denote different constants, even in a sequence of relations.

For $r, k \in \mathbb{N}$ we let $P_k^{r,X} \in \mathscr{L}(C(Q,X))$ be X-valued composite tensor product Lagrange interpolation of degree r with respect to the partition of $[0,1]^d$ into k^d subcubes of sidelength k^{-1} of disjoint interior, see [2]. Given $r \in \mathbb{N}_0$ and $d \in \mathbb{N}$, there are constants $c_1, c_2 > 0$ such that for all Banach spaces X and all $k \in \mathbb{N}$

$$\sup_{f \in B_{C^r(Q,X)}} \|f - P_k^{r,X} f\|_{C(Q,X)} \le c_2 k^{-r} \tag{3}$$

(see [2]).

3 Banach space valued integration

Let X be a Banach space, $r \in \mathbb{N}_0$, and let the integration operator $S^X : C(Q,X) \to X$ be given by

$$S^X f = \int_Q f(t)dt.$$

We will work in the setting of information-based complexity theory, see [12, 10, 11]. Below $e_n^{\mathrm{det}}(S^X, B_{C^r(Q,X)})$ and $e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)})$ denote the n-th minimal error of S^X on $B_{C^r(Q,X)}$ in the deterministic, respectively randomized setting, that is, the minimal possible error among all deterministic, respectively randomized algorithms, approximating S^X on $B_{C^r(Q,X)}$ that use at most n values of the input function f. The precise notions are recalled in the appendix. The following was shown in [2].

Theorem 1. *Let $r \in \mathbb{N}_0$ and $1 \le p \le 2$. Then there are constants $c_{1-4} > 0$ such that for all Banach spaces X and $n \in \mathbb{N}$ the following holds. The deterministic n-th minimal error satisfies*

$$c_1 n^{-r/d} \le e_n^{\mathrm{det}}(S^X, B_{C^r(Q,X)}) \le c_2 n^{-r/d}.$$

Moreover, if X is of type p and p_X is the supremum of all p_1 such that X is of type p_1, then the randomized n-th minimal error fulfills

$$c_3 n^{-r/d-1+1/p_X} \le e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \le c_4 \tau_p(X) n^{-r/d-1+1/p}.$$

As a consequence, we obtain

Corollary 1. *Let $r \in \mathbb{N}_0$ and $1 \le p \le 2$. Then the following are equivalent:*
(i) X is of type p_1 for all $p_1 < p$.
(ii) For each $p_1 < p$ there is a constant $c > 0$ such that for all $n \in \mathbb{N}$

$$e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \le c n^{-r/d-1+1/p_1}.$$

The main result of the present paper is the following

Theorem 2. *Let $1 \le p \le 2$ and $r \in \mathbb{N}_0$. Then there are constants $c_1, c_2 > 0$ such that for all Banach spaces X and all $n \in \mathbb{N}$*

$$c_1 n^{r/d+1-1/p} e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \le \sigma_{p,n}(X) \le c_2 \max_{1\le k\le n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}). \tag{4}$$

This allows to sharpen Corollary 1 in the following way.

Corollary 2. *Let $r \in \mathbb{N}_0$ and $1 \le p \le 2$. Then the following are equivalent:*
(i) X is of equal norm type p.
(ii) There is a constant $c > 0$ such that for all $n \in \mathbb{N}$

$$e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \le c n^{-r/d-1+1/p}.$$

Recall from the preliminaries that the conditions in the corollary are also equivalent to
(iii) X is of type 2 if $p = 2$ and of weak type p if $1 < p < 2$, respectively.

For the proof of Theorem 2 we need a number of auxiliary results. The following lemma is a slight modification of Prop. 9.11 of [7], with essentially the same proof, which we include for the sake of completeness.

Lemma 1. *Let $1 \le p \le 2$. Then there is a constant $c > 0$ such that for each Banach space X, each $n \in \mathbb{N}$ and each sequence of independent, essentially bounded, mean zero X-valued random variables $(\eta_i)_{i=1}^n$ on some probability space $(\Omega, \Sigma, \mathbb{P})$ the following holds:*

$$\left(\mathbb{E}\left\|\sum_{i=1}^n \eta_i\right\|^p\right)^{1/p} \le c\sigma_{p,n}(X) n^{1/p} \max_{1\le i\le n} \|\eta_i\|_{L_\infty(\Omega,\mathbb{P},X)}.$$

Proof. Let $(\varepsilon_i)_{i=1}^n$ be independent, symmetric Bernoulli random variables on some probability space $(\Omega', \Sigma', \mathbb{P}')$ different from $(\Omega, \Sigma, \mathbb{P})$. Considering $(\eta_i)_{i=1}^n$ and

$(\varepsilon_i)_{i=1}^n$ as random variables on the product probability space, we denote the expectation with respect to $\mathbb{P}'$ by $\mathbb{E}'$ (and the expectation with respect to $\mathbb{P}$, as before, by $\mathbb{E}$). Using Lemma 6.3 of [7] and (2), we get

$$\begin{aligned}\left(\mathbb{E}\Big\|\sum_{i=1}^n \eta_i\Big\|^p\right)^{1/p} &\le 2\left(\mathbb{E}\,\mathbb{E}'\Big\|\sum_{i=1}^n \varepsilon_i\eta_i\Big\|^p\right)^{1/p}\\ &\le 2\sigma_{p,n}(X)n^{1/p}\left(\mathbb{E}\max_{1\le i\le n}\|\eta_i\|^p\right)^{1/p}\\ &\le 2\sigma_{p,n}(X)n^{1/p}\max_{1\le i\le n}\|\eta_i\|_{L_\infty(\Omega,\mathbb{P},X)}.\end{aligned}$$

□

Next we introduce an algorithm for the aproximation of $S^X f$. Let $n \in \mathbb{N}$ and let $\xi_i : \Omega \to Q$ $(i = 1, \dots, n)$ be independent random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$ uniformly distributed on Q. Define for $f \in C(Q, X)$

$$A_{n,\omega}^{0,X} f = \frac{1}{n}\sum_{i=1}^n f(\xi_i(\omega)) \tag{5}$$

and, if $r \ge 1$, put $k = \lceil n^{1/d} \rceil$ and

$$A_{n,\omega}^{r,X} f = S^X(P_k^{r,X} f) + A_{n,\omega}^{0,X}(f - P_k^{r,X} f). \tag{6}$$

These are the Banach space valued versions of the standard Monte Carlo method $(r = 0)$ and the Monte Carlo method with separation of the main part $(r \ge 1)$. The following extends the second part of Proposition 1 of [2].

Proposition 1. *Let $r \in \mathbb{N}_0$ and $1 \le p \le 2$. Then there is a constant $c > 0$ such that for all Banach spaces X, $n \in \mathbb{N}$, and $f \in C^r(Q, X)$*

$$\left(\mathbb{E}\,\|S^X f - A_{n,\omega}^{r,X} f\|^p\right)^{1/p} \le c\sigma_{p,n}(X) n^{-r/d-1+1/p}\|f\|_{C^r(Q,X)}. \tag{7}$$

Proof. Let us first consider the case $r = 0$. Let $f \in C(Q, X)$ and put

$$\eta_i(\omega) = \int_Q f(t)dt - f(\xi_i(\omega)).$$

Clearly, $\mathbb{E}\,\eta_i(\omega) = 0$,

$$S^X f - A_{n,\omega}^{0,X} f = \frac{1}{n}\sum_{i=1}^n \eta_i(\omega)$$

and

$$\|\eta_i(\omega)\| \le 2\|f\|_{C(Q,X)}.$$

An application of Lemma 1 gives (7). If $r \ge 1$, we have

$$S^X f - A_{n,\omega}^{r,X} f = S^X(f - P_k^{r,X} f) - A_{n,\omega}^{0,X}(f - P_k^{r,X} f)$$

and the result follows from (3) and the case $r = 0$. □

Lemma 2. *Let $1 \le p \le 2$. Then there are constants $c > 0$ and $0 < \gamma < 1$ such that for each Banach space X, each $n \in \mathbb{N}$, and $(x_i)_{i=1}^n \subset X$ there is a subset $I \subseteq \{1, \dots, n\}$ with $|I| \ge \gamma n$ and*

$$\mathbb{E}\Big\| \sum_{i \in I} \varepsilon_i x_i \Big\| \le c n^{1/p} \|(x_i)\|_{\ell_\infty^n(X)} \max_{1 \le k \le n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}).$$

Proof. Since for $n \in \mathbb{N}$

$$\max_{1 \le k \le n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \ge e_1^{\mathrm{ran}}(S^{\mathbb{K}}, B_{C^r(Q,\mathbb{K})}) > 0,$$

the statement is trivial for $n < 8^d$. Therefore we can assume $n \ge 8^d$. Clearly, we can also assume $\|(x_i)\|_{\ell_\infty^n(X)} > 0$. Let $m \in \mathbb{N}$ be such that

$$m^d \le n < (m+1)^d, \tag{8}$$

hence

$$m \ge 8. \tag{9}$$

Let ψ be an infinitely differentiable function on $\mathbb{R}^d$ such that $\psi(t) > 0$ for $t \in (0,1)^d$ and $\operatorname{supp} \psi \subset [0,1]^d$. Let $(Q_i)_{i=1}^{m^d}$ be the partition of Q into closed cubes of side length m^{-1} of disjoint interior, let t_i be the point in Q_i with minimal coordinates and define $\psi_i \in C(Q)$ by

$$\psi_i(t) = \psi(m(t - t_i)) \quad (i = 1, \dots, m^d).$$

It is easily verified that there is a constant $c_0 > 0$ such that for all $(\alpha_i)_{i=1}^{m^d} \in [-1,1]^{m^d}$

$$\Big\| \sum_{i=1}^{m^d} \alpha_i x_i \psi_i \Big\|_{C^r(Q,X)} \le c_0 m^r \|(x_i)\|_{\ell_\infty^n(X)}.$$

Setting

$$f_i = c_0^{-1} m^{-r} \|(x_i)\|_{\ell_\infty^n(X)}^{-1} x_i \psi_i$$

it follows that

$$\sum_{i=1}^{m^d} \alpha_i f_i \in B_{C^r(Q,X)} \qquad \text{for all} \quad (\alpha_i)_{i=1}^{m^d} \in [-1,1]^{m^d}.$$

Moreover, with $\sigma = \int_Q \psi(t)dt$ we have

$$\Big\| \sum_{i=1}^{m^d} \alpha_i S^X f_i \Big\| = c_0^{-1} m^{-r} \|(x_i)\|_{\ell_\infty^n(X)}^{-1} \Big\| \sum_{i=1}^{m^d} \alpha_i x_i \int_Q \psi_i(t)dt \Big\|$$

$$= \quad c_0^{-1} \sigma m^{-r-d} \|(x_i)\|_{\ell_\infty^n(X)}^{-1} \Big\| \sum_{i=1}^{m^d} \alpha_i x_i \Big\|.$$

Next we use Lemma 5 and 6 of [3] with $K = X$ (although stated for $K = \mathbb{R}$, Lemma 6 is easily seen to hold for $K = X$, as well) to obtain for all $l \in \mathbb{N}$ with $l < m^d/4$

$$\begin{aligned} e_l^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) &\geq \frac{1}{4} \min_{I \subseteq \{1,\dots,m^d\}, |I| \geq m^d - 4l} \mathbb{E} \Big\| \sum_{i\in I} \varepsilon_i S^X f_i \Big\| \\ &\geq c m^{-r-d} \|(x_i)\|^{-1}_{\ell_\infty^n(X)} \mathbb{E} \Big\| \sum_{i \in I} \varepsilon_i x_i \Big\|. \end{aligned}$$

We put $l = \lfloor m^d/8 \rfloor$. Then

$$m^d/16 < l \leq m^d/8. \tag{10}$$

Indeed, by (9) the left-hand inequality clearly holds for $m^d < 16$, while for $m^d \geq 16$ we get $\lfloor m^d/8 \rfloor > m^d/8 - 1 \geq m^d/16$. We conclude that there is an $I \subseteq \{1, \dots, m^d\}$ with $|I| \geq m^d - 4l \geq m^d/2$ and

$$\begin{aligned} \mathbb{E} \Big\| \sum_{i\in I} \varepsilon_i x_i \Big\| &\leq c m^{r+d} \|(x_i)\|_{\ell_\infty^n(X)} e_l^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \\ &\leq c m^{r+d} l^{-r/d+1/p-1} \|(x_i)\|_{\ell_\infty^n(X)} \max_{1\leq k\leq n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \\ &\leq c n^{1/p} \|(x_i)\|_{\ell_\infty^n(X)} \max_{1\leq k\leq n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}), \end{aligned}$$

where we used (8) and (10). Finally, (8) and (9) give

$$|I| \geq m^d/2 \geq \frac{m^d}{2(m+1)^d} n \geq \frac{8^d}{2 \cdot 9^d} n.$$

□

Proof of Theorem 2. The left-hand inequality of (4) follows directly from Proposition 1, since the number of function values involved in $A_{n,\omega}^{r,X}$ is bounded by $ck^d + n \leq cn$, see also (16).

To prove the right-hand inequality of (4), let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$. We construct by induction a partition of $K = \{1, \dots, n\}$ into a sequence of disjoint subsets $(I_l)_{l=1}^{l^*}$ such that for $1 \leq l \leq l^*$

$$|I_l| \geq \gamma \Big| K \setminus \bigcup_{j<l} I_j \Big| \tag{11}$$

and

$$\begin{aligned} &\mathbb{E} \Big\| \sum_{i\in I_l} \varepsilon_i x_i \Big\| \\ &\leq c \Big| K \setminus \bigcup_{j<l} I_j \Big|^{1/p} \|(x_i)\|_{\ell_\infty^n(X)} \max_{1\leq k\leq n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}), \end{aligned} \tag{12}$$

where c and γ are the constants from Lemma 2. For $l=1$ the existence of an I_1 satisfying (11–12) follows directly from Lemma 2. Now assume that we already have a sequence of disjoint subsets $(I_l)_{l=1}^m$ of K satisfying (11–12). If

$$J := K \setminus \bigcup_{j\le m} I_j \ne \emptyset,$$

we apply Lemma 2 to $(x_i)_{i\in J}$ to find $I_{m+1}\subseteq J$ with

$$|I_{m+1}| \ge \gamma |J| \tag{13}$$

and

$$\begin{aligned} &\mathbb{E}\Big\| \sum_{i\in I_{m+1}} \varepsilon_i x_i \Big\| \\ &\le\ c|J|^{1/p} \|(x_i)_{i\in J}\|_{\ell_\infty(J,X)} \max_{1\le k\le |J|} k^{r/d+1-1/p} e_k^{\rm ran}(S^X, B_{C^r(Q,X)}). \end{aligned}\tag{14}$$

Observe that for $l=m+1$, (13) is just (11) and (14) implies (12). Furthermore, (11) implies

$$\Big|K \setminus \bigcup_{j\le l} I_j\Big| \le (1-\gamma)\Big|K\setminus \bigcup_{j\le l-1} I_j\Big|$$

and therefore

$$\Big|K \setminus \bigcup_{j\le l} I_j\Big| \le (1-\gamma)^l n. \tag{15}$$

It follows that the process stops with $K=\bigcup_{j\le l} I_j$ for a certain $l=l^*\in\mathbb{N}$. This completes the construction.

Using the equivalence of moments (Theorem 4.7 of [7]), we get from (12) and (15)

$$\begin{aligned} &\left(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^p\right)^{1/p} \\ &\le\ c\,\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\| \le c\sum_{l=1}^{l^*} \mathbb{E}\Big\|\sum_{i\in I_l}\varepsilon_i x_i\Big\| \\ &\le\ cn^{1/p}\|(x_i)\|_{\ell_\infty^n(X)} \max_{1\le k\le n} k^{r/d+1-1/p} e_k^{\rm ran}(S^X, B_{C^r(Q,X)}) \sum_{l=1}^{l^*} (1-\gamma)^{(l-1)/p}. \end{aligned}$$

This gives the upper bound of (4).

$\square$

Let us mention that results analogous to Theorem 2 and Corollary 2 above also hold for Banach space valued indefinite integration (see [2] for the definition) and for the solution of initial value problems for Banach space valued ordinary differential equations [5]. Indeed, an inspection of the respective proofs together with Lemma 1 of the present paper shows that Proposition 2 of [2] also holds with $\tau_p(X)$ replaced by $\sigma_{p,n}(X)$, and similarly Proposition 3.4 of [5]. Moreover, in both papers the lower bounds on $e_n^{\rm ran}$ are obtained by reduction to (definite) integration and thus the righ-hand side inequality of (4) carries over directly.

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4 Appendix

In this appendix we recall some basic notions of information-based complexity – the framework we used above. We refer to [10, 12] for more on this subject and to [3, 4] for the particular notation applied here. First we introduce the class of deterministic adaptive algorithms of varying cardinality $\mathcal{A}^{\det}(C(Q,X),X)$. It consists of tuples $A = ((L_i)_{i=1}^\infty, (\varrho_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty)$, with $L_1 \in Q$, $\varrho_0 \in \{0,1\}$, $\varphi_0 \in X$ and

$$L_i : X^{i-1} \to Q \quad (i = 2,3,\dots), \quad \varrho_i : X^i \to \{0,1\}, \; \varphi_i : X^i \to X \quad (i = 1,2,\dots)$$

being arbitrary mappings. To each $f \in C(Q,X)$, we associate a sequence $(t_i)_{i=1}^\infty$ with $t_i \in Q$ as follows:

$$t_1 = L_1, \quad t_i = L_i(f(t_1),\dots,f(t_{i-1})) \quad (i \geq 2).$$

Define $\operatorname{card}(A,f)$, the cardinality of A at input f, to be 0 if $\varrho_0 = 1$. If $\varrho_0 = 0$, let $\operatorname{card}(A,f)$ be the first integer $n \geq 1$ with $\varrho_n(f(t_1),\dots,f(t_n)) = 1$, if there is such an n, and $\operatorname{card}(A,f) = +\infty$ otherwise. For $f \in C(Q,X)$ with $\operatorname{card}(A,f) < \infty$ we define the output Af of algorithm A at input f as

$$Af = \begin{cases} \varphi_0 & \text{if} \quad n = 0 \\ \varphi_n(f(t_1),\dots,f(t_n)) & \text{if} \quad n \geq 1. \end{cases}$$

Let $r \in \mathbb{N}_0$. Given $n \in \mathbb{N}_0$, we let $\mathcal{A}_n^{\det}(B_{C^r(Q,X)},X)$ be the set of those $A \in \mathcal{A}^{\det}(C(Q,X),X)$ for which

$$\max_{f \in B_{C^r(Q,X)}} \operatorname{card}(A,f) \leq n.$$

The error of $A \in \mathcal{A}_n^{\det}(B_{C^r(Q,X)},X)$ as an approximation of S^X is defined as

$$e(S^X, A, B_{C^r(Q,X)}) = \sup_{f \in B_{C^r(Q,X)}} \|S^X f - Af\|.$$

The deterministic n-th minimal error of S^X is defined for $n \in \mathbb{N}_0$ as

$$e_n^{\det}(S^X, B_{C^r(Q,X)}) = \inf_{A \in \mathcal{A}_n^{\det}(B_{C^r(Q,X)})} e(S^X, A, B_{C^r(Q,X)}).$$

It follows that no deterministic algorithm that uses at most n function values can have a smaller error than $e_n^{\det}(S^X, B_{C^r(Q,X)})$.

Next we introduce the class of randomized adaptive algorithms of varying cardinality $\mathcal{A}_n^{\mathrm{ran}}(B_{C^r(Q,X)},X)$, consisting of tuples $A = ((\Omega,\Sigma,\mathbb{P}),(A_\omega)_{\omega\in\Omega})$, where $(\Omega,\Sigma,\mathbb{P})$ is a probability space, $A_\omega \in \mathcal{A}^{\det}(C(Q,X),X)$ for all $\omega \in \Omega$, and for each $f \in B_{C^r(Q,X)}$ the mapping $\omega \in \Omega \to \operatorname{card}(A_\omega,f)$ is Σ-measurable and satisfies $\mathbb{E}\operatorname{card}(A_\omega,f) \leq n$. Moreover, the mapping $\omega \in \Omega \to A_\omega f \in X$ is Σ-to-Borel measurable and essentially separably valued, i.e., there is a separable subspace $X_0 \subseteq X$ such that $A_\omega f \in X_0$ for $\mathbb{P}$-almost all $\omega \in \Omega$. The error of $A \in \mathcal{A}_n^{\mathrm{ran}}(C(Q,X),X)$ in approximating S^X on $B_{C^r(Q,X)}$ is defined as

$$e(S^X, A, B_{C^r(Q,X)}) = \sup_{f \in B_{C^r(Q,X)}} \mathbb{E}\, \|S^X f - A_\omega f\|,$$

and the randomized n-th minimal error of S^X as

$$e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) = \inf_{A \in \mathcal{A}_n^{\mathrm{ran}}(B_{C^r(Q,X)})} e(S^X, A, B_{C^r(Q,X)}).$$

Consequently, no randomized algorithm that uses (on the average) at most n function values has an error smaller than $e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}, X)$.

Define for $\varepsilon > 0$ the information complexity as

$$n_\varepsilon^{\mathrm{ran}}(S, B_{C^r(Q,X)}) = \min\{n \in \mathbb{N}_0 : \ e_n^{\mathrm{ran}}(S, B_{C^r(Q,X)}) \le \varepsilon\},$$

if there is such an n, and $n_\varepsilon^{\mathrm{ran}}(S, B_{C^r(Q,X)}) = +\infty$, if there is no such n. Thus, if $n_\varepsilon^{\mathrm{ran}}(S, B_{C^r(Q,X)}) < \infty$, it follows that any algorithm with error $\le \varepsilon$ needs at least $n_\varepsilon^{\mathrm{ran}}(S, B_{C^r(Q,X)})$ function values, while $n_\varepsilon^{\mathrm{ran}}(S, B_{C^r(Q,X)}) = +\infty$ means that no algorithm at all has error $\le \varepsilon$. The information complexity is essentially the inverse function of the n-th minimal error. So determining the latter means determining the information complexity of the problem.

Let us also mention the subclasses consisting of quadrature formulas. Let $n \ge 1$. A mapping $A : C(Q,X) \to X$ is called a deterministic quadrature formula with n nodes, if there are $t_i \in Q$ and $a_i \in \mathbb{K}$ $(1 \le i \le n)$ such that

$$Af = \sum_{i=1}^n a_i f(t_i) \quad (f \in C(Q,X)).$$

In terms of the definition of $\mathcal{A}^{\mathrm{det}}(C(Q,X),X)$ this means that the respective functions L_i and ϱ_i are constant, $\varrho_0 = \varrho_1 = \cdots = \varrho_{n-1} = 0$, $\varrho_n = 1$, and φ_n has the form $\varphi_n(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i$. Clearly, $A \in \mathcal{A}_n^{\mathrm{det}}(B_{C^r(Q,X)}, X)$.

A tupel $A = ((\Omega, \Sigma, \mathbb{P}), (A_\omega)_{\omega \in \Omega})$ is called a randomized quadrature with n nodes if there exist random variables $t_i : \Omega \to Q$ and $a_i : \Omega \to \mathbb{K}$ $(1 \le i \le n)$ with

$$A_\omega f = \sum_{i=1}^n a_i(\omega) f(t_i(\omega)) \quad (f \in C(Q,X),\, \omega \in \Omega).$$

For each such A we have $A \in \mathcal{A}_n^{\mathrm{ran}}(B_{C^r(Q,X)}, X)$. Finally we note that the algorithms $A_{n,\omega}^{r,X}$ defined in (5) and (6) are quadratures. Indeed, for $A_{n,\omega}^{0,X}$ given by (5) this is obvious. For $r \ge 1$ we represent $P_k^{r,X} \in \mathscr{L}(C(Q,X))$ as

$$P_k^{r,X} f = \sum_{j=1}^M f(u_j) \psi_j(t)$$

with $M \le ck^d$, $u_j \in Q$, $\psi_j \in C(Q)$ $(1 \le i \le M)$, and obtain, setting $b_j = \int_Q \psi_j(t)dt$,

$$
\begin{aligned}
A_{n,\omega}^{r,X} f &= S^X(P_k^{r,X} f) + A_{n,\omega}^{0,X}(f - P_k^{r,X} f) \\
&= \sum_{j=1}^{M} b_j f(u_j) + \frac{1}{n} \sum_{i=1}^{n} \Big(f(\xi_i(\omega)) - \Big(P_k^{r,X} f \Big) (\xi_i(\omega)) \Big) \\
&= \sum_{j=1}^{M} b_j f(u_j) + \frac{1}{n} \sum_{i=1}^{n} f(\xi_i(\omega)) - \sum_{j=1}^{M} \left(\frac{1}{n} \sum_{i=1}^{n} \psi_j(\xi_i(\omega)) \right) f(u_j). \quad (16)
\end{aligned}
$$