Complexity of parametric initial value problems for systems of ODEs

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Abstract

We study the approximate solution of initial value problems for parameter dependent finite or infinite systems of scalar ordinary differential equations (ODEs). Both the deterministic and the randomized setting is considered, with input data from various smoothness classes. We study deterministic and Monte Carlo multilevel algorithms and derive convergence rates. Moreover, we prove their optimality by showing matching (in some limit cases up to logarithmic factors) lower bounds and settle this way the complexity. Comparisons between the deterministic and randomized setting are given, as well.

1 Introduction

This paper is a continuation of [3], where we considered the complexity of parameter dependent ODEs in Banach spaces. Here we study initial value problems for parameter dependent finite or infinite scalar systems of ODEs. We consider both the deterministic and the randomized setting, and various classes of input functions.

We apply the algorithm and its analysis from [3] to scalar systems. The rates obtained in [3] for general Banach spaces were sharp up to an arbitrary small gap in the exponent. Using techniques from [4], in the present study we derive more precise estimates – we determine the order, in some limit cases up to logarithmic factors. Moreover, while in [3] only classes defined on the whole space were considered, in this part more general local classes are studied. Finally, based on finite dimensional estimates from [8] and [4], we prove lower bounds and obtain the complexity.

The paper is organized as follows. Section 2 contains preliminaries. Convergence rates are derived in Section 3. In Section 4 we prove lower bounds and present the complexity analysis. We also discuss one special case of the considered classes – functions with dominating mixed smoothness – and give comparisons between the deterministic and the randomized setting. For a more extended bibliography as well as further background material we refer to [3].

2 Preliminaries

The main goal of this paper is the study of parameter dependent finite systems of scalar ODEs, that is, in the terminology of [3], we have $Z = \mathbb{R}^d$ for some $d \in \mathbb{N}$. However, we will still consider the more general case $Z = H$, where H is any Hilbert space over the reals. This way we also include infinite systems of scalar ODEs.

We start with the definition of the needed function classes. The functions considered in [3] where assumed to possess certain smoothness properties on all of H. We now introduce larger, local classes. Let $d_0 \in \mathbb{N}$, $Q = [0,1]^{d_0}$. Let B_H denote the closed and B_H^0 the open unit ball of H. Given $r_0, r \in \mathbb{N}_0$, $0 \le \rho \le 1$, $\lambda_1, \kappa, L > 0$, and a real Hilbert space H, we define the following class $\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\rho}(Q \times [a,b] \times \lambda_1 B_H^0, H; \kappa, L)$ of continuous functions $f : Q \times [a,b] \times \lambda_1 B_H^0 \to H$ having for $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}_0^3$ with $\alpha_0 \le r_0$, $\alpha_1 \le r$, and $\alpha_0 + \alpha_1 + \alpha_2 \le r_0 + r$ continuous partial (Fréchet) derivatives $\frac{\partial^{|\alpha|} f(s,t,z)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}}$ satisfying for $s \in Q$, $t \in [a,b]$, $z \in \lambda_1 B_H^0$

$$\left\| \frac{\partial^{|\alpha|} f(s,t,z)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} \right\| \le \kappa, \tag{1}$$

for $s \in Q$, $t_1, t_2 \in [a,b]$, $z_1, z_2 \in \lambda_1 B_H^0$

$$\left\| \frac{\partial^{|\alpha|} f(s,t_1,z_1)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} - \frac{\partial^{|\alpha|} f(s,t_2,z_2)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} \right\| \le \kappa |t_1 - t_2|^\rho + \kappa \|z_1 - z_2\|^\rho, \tag{2}$$

and for $\alpha = (\alpha_0, 0, \alpha_2)$ with $\alpha_0 + \alpha_2 \le r_0$, $s \in Q$, $t \in [a,b]$, $z_1, z_2 \in \lambda_1 B_H^0$

$$\left\| \frac{\partial^{|\alpha|} f(s,t,z_1)}{\partial s^{\alpha_0} \partial z^{\alpha_2}} - \frac{\partial^{|\alpha|} f(s,t,z_2)}{\partial s^{\alpha_0} \partial z^{\alpha_2}} \right\| \le L \|z_1 - z_2\|. \tag{3}$$

Let $\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\rho}(Q \times [a,b] \times H, H; \kappa, L)$ denote the class defined in the same way as above, just with $\lambda_1 B_H^0$ replaced by H. We write $\mathscr{C}_{\mathrm{Lip}}^{r,\rho}([a,b] \times \lambda_1 B_H^0, H; \kappa, L)$ for the subclass of $\mathscr{C}_{\mathrm{Lip}}^{0,r,\rho}(Q \times [a,b] \times \lambda_1 B_H^0, H; \kappa, L)$ consisting of functions not depending on s. In the sequel we also use the notation f_s, where $s \in Q$, for the function $f(s, \cdot, \cdot)$ from $[a,b] \times \lambda_1 B_H^0$ to H.

Given $f \in \mathscr{C}_{\mathrm{Lip}}^{r_0,r,\rho}(Q \times [a,b] \times \lambda_1 B_H^0, H; \kappa, L)$ and $u_0 \in \lambda_1 B_H^0$, we consider the parameter dependent initial value problem

$$\frac{d}{dt} u(s,t) = f(s,t,u(s,t)) \quad (s \in Q, t \in [a,b]), \tag{4}$$

$$u(s,a) = u_0(s) \quad (s \in Q). \tag{5}$$

A function $u : Q \times [a,b] \to H$ is called a solution if for each $s \in Q$, $u(s,t)$ is continuously differentiable as a function of t, $u(s,t) \in \lambda_1 B_H^0$ for all $s \in Q, t \in [a,b]$, and (4–5) are satisfied.

Next we recall the algorithm developed and studied for the scalar case in [1] and for the Banach space valued case in [7, 3]. It produces an approximate solution to the non-parametric version of (4–5), that is, $f \in \mathscr{C}_{\mathrm{Lip}}^{r,\rho}([a,b] \times \lambda_1 B_H^0, H; \kappa, L)$ and $u_0 \in H$ do not depend on s. We have to modify the definition since, in contrast to [1, 7, 3], here the algorithm needs not always be defined.

Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, and put $h = (b-a)/n$, $t_k = a + kh$ $(k = 0, 1, \dots, n)$. Furthermore, for $0 \le k \le n-1$ and $1 \le j \le m$ let $P_{k,j}$ be the operator of Lagrange interpolation of degree j on the equidistant grid $t_{k,j,i} = t_k + ih/j$ $(i = 0, \dots, j)$ on $[t_k, t_{k+1}]$. Let $\xi_1, \dots, \xi_n$ be independent random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$ such that ξ_k is uniformly distributed on $[t_{k-1}, t_k]$ and

$$\{(\xi_1(\omega), \dots, \xi_n(\omega)) : \omega \in \Omega\} = [t_0, t_1] \times \cdots \times [t_{n-1}, t_n].$$

We define $(u_k)_{k=1}^n \subset H$ and H-valued polynomials $p_{k,j}(t)$ for $k = 0, \dots, n-1$, $j = 0, \dots, m$ by induction. Let $0 \le k \le n-1$, suppose u_k is already defined and

$$u_k \in \lambda_1 B_H^0. \tag{6}$$

Then we put

$$p_{k,0}(t) \;=\; u_k + f(t_k, u_k)(t - t_k) \quad (t \in [t_k, t_{k+1}]). \tag{7}$$

Furthermore, if $m \ge 1$, $0 \le j < m$, $p_{k,j}$ is already defined, and

$$p_{k,j}(t_{k,j+1,i}) \in \lambda_1 B_H^0 \quad (i = 0, \dots, j+1), \tag{8}$$

then we set

$$q_{k,j} \;=\; \left(f(t_{k,j+1,i}, p_{k,j}(t_{k,j+1,i}))\right)_{i=0}^{j+1} \tag{9}$$

and

$$p_{k,j+1}(t) \;=\; u_k + \int_{t_k}^{t} \left(P_{k,j+1} q_{k,j}\right)(\tau)\, d\tau. \tag{10}$$

Finally, if

$$p_{k,m}(t) \in \lambda_1 B_H^0 \quad (t \in [t_k, t_{k+1}]), \tag{11}$$

we define

$$u_{k+1} = p_{k,m}(t_{k+1}) + h\left(f(\xi_{k+1}, p_{k,m}(\xi_{k+1})) - p'_{k,m}(\xi_{k+1})\right). \tag{12}$$

Now let $B([a,b], H)$ denote the space of all H-valued, bounded on $[a,b]$ functions, equipped with the supremum norm. We define $v \in B([a,b], H)$ by

$$v(t) = \begin{cases} p_{k,m}(t) & \text{if} \quad t \in [t_k, t_{k+1}) \quad \text{and} \quad 0 \le k \le n-1, \\ u_n & \text{if} \quad t = t_n. \end{cases} \tag{13}$$

For $\omega \in \Omega$ fixed, let

$$A^m_{n,\omega} : C^{r,\rho}_{\mathrm{Lip}}([a,b] \times \lambda_1 B^0_H, H; \kappa, L) \times H \to B([a,b], H)$$

denote the resulting mapping, that is,

$$A^m_{n,\omega}(f, u_0) = v. \tag{14}$$

We say that $A^m_{n,\omega}(f, u_0)$ is defined on $[a,b] \times \lambda_1 B^0_H$ (or, shortly: defined), if this definition goes through till (13), that is, (6), (8), (11) are satisfied for all $0 \le k \le n-1$ and, if $m \ge 1$, for $0 \le j \le m-1$. If for some ω and some k, any of the conditions (6), (8), (11) is violated, we leave $A^m_{n,\omega}(f, u_0)$ undefined. Note that for $m = 0$ we have

$$\begin{aligned} p_{k,0}(t) &= u_k + f(t_k, u_k)(t - t_k) \qquad (t \in [t_k, t_{k+1}],\ 0 \le k \le n-1), \qquad (15)\\ u_{k+1} &= u_k + h f(\xi_{k+1}, p_{k,0}(\xi_{k+1})) \qquad (0 \le k \le n-1). \qquad (16) \end{aligned}$$

Given also $\sigma > 0$ and $\lambda_0 > 0$ with $\lambda_0 < \lambda_1$, let $\mathscr{F}$ be the class of all

$$\begin{aligned} (f, u_0) \in \Big(&\mathscr{C}^{0,r,\rho}_{\mathrm{Lip}}(Q \times [a,b] \times \lambda_1 B^0_H, H; \kappa, L) \\ &\cap \mathscr{C}^{r_0,r_1,\rho_1}_{\mathrm{Lip}}(Q \times [a,b] \times \lambda_1 B^0_H, H; \kappa, L)\Big) \times \sigma B_{C^{r_0}(Q,H)} \end{aligned} \tag{17}$$

such that the parameter dependent initial value problem (4–5) has a solution $u(s,t)$ with

$$\sup_{s \in Q,\, t \in [a,b]} \|u(s,t)\| \le \lambda_0, \tag{18}$$

and moreover, if $r = \rho = r_1 = \rho_1 = 0$, then for all $n \in \mathbb{N}$, $\omega \in \Omega$, $s \in Q$, $A^0_{n,\omega}(f_s, u_0(s))$ is defined on $[a,b] \times \lambda_1 B^0_H$ and

$$\sup_{s \in Q} \left\| A^0_{n,\omega}(f_s, u_0(s)) \right\|_{B([a,b],H)} \le \lambda_0. \tag{19}$$

Note that if

$$\sigma + \kappa(b-a) \le \lambda_0,$$

then (18) and, in the case $r = \rho = r_1 = \rho_1 = 0$, also (19) are automatically satisfied, that is, we have

$$\begin{aligned} \mathscr{F} = \Big(&\mathscr{C}^{0,r,\rho}_{\mathrm{Lip}}(Q \times [a,b] \times \lambda_1 B^0_H, H; \kappa, L) \\ &\cap \mathscr{C}^{r_0,r_1,\rho_1}_{\mathrm{Lip}}(Q \times [a,b] \times \lambda_1 B^0_H, H; \kappa, L)\Big) \times \sigma B_{C^{r_0}(Q,H)}. \end{aligned}$$

The solution operator

$$\mathscr{S} : \mathscr{F} \to B(Q \times [a,b], H) \tag{20}$$

is given for $(f, u_0) \in \mathscr{F}$ by $\mathscr{S}(f, u_0) = u$.

The following multilevel algorithm for the approximate solution of the parametric problem (4–5) was already introduced in [3]. Let $l_0, l_1 \in \mathbb{N}_0$, $l_0 \le l_1$, $n_{l_0}, \dots, n_{l_1} \in \mathbb{N}$, $\omega \in \Omega$, and set

$$\begin{aligned}\mathscr{A}_\omega(f, u_0) &= P_{l_0}\left(\left(A^r_{n_{l_0},\omega}(f_s, u_0(s))\right)_{s\in\Gamma_{l_0}}\right) \\ &\quad + \sum_{l=l_0+1}^{l_1} (P_l - P_{l-1})\left(\left(A^{r_1}_{n_l,\omega}(f_s, u_0(s))\right)_{s\in\Gamma_l}\right). \end{aligned} \tag{21}$$

Here P_l is H-valued composite with respect to the partition of Q into cubes of sidelength 2^{-l} tensor product Lagrange interpolation of degree $\max(r_0, 1)$. Furthermore, Γ_l is the equidistant grid on Q of meshsize $(\max(r_0, 1))^{-1}2^{-l}$. The algorithms $A^r_{n_{l_0},\omega}(f_s, u_0(s))$ $(s \in \Gamma_{l_0})$ and $A^{r_1}_{n_l,\omega}(f_s, u_0(s))$ $(s \in \Gamma_l, l_0 < l \le l_1)$ are given by (6–14). We say that $\mathscr{A}_\omega(f, u_0)$ is defined, if $A^r_{n_{l_0},\omega}(f_s, u_0(s))$ $(s \in \Gamma_{l_0})$ and $A^{r_1}_{n_l,\omega}(f_s, u_0(s))$ $(s \in \Gamma_l, l_0 < l \le l_1)$ are defined.

If $\mathrm{card}(\mathscr{A}_\omega)$ denotes the number of function evaluations involved in $\mathscr{A}_\omega$, we have

$$\mathrm{card}(\mathscr{A}_\omega) \le c \sum_{l=l_0}^{l_1} n_l 2^{d_0 l}. \tag{22}$$

Furthermore, the number of arithmetic operations (including addition and multiplication by scalars of elements in H) of $\mathscr{A}_\omega$ is bounded from above by $c\,\mathrm{card}(\mathscr{A}_\omega)$ for some $c > 0$.

3 Error estimates

To formulate the first result, we introduce the following functions. For $n \ge 2$ we set (throughout the paper log means $\log_2$)

$$\theta_1(n) = \begin{cases} 1 & \text{if } \left(\frac{r_0}{d_0} \ne r_1 + \rho_1\right) \vee \left(\frac{r_0}{d_0} = r_1 + \rho_1 = 0\right) \\ (\log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0} = r_1 + \rho_1 > 0 \end{cases} \tag{23}$$

and for $n \ge 3$

$$\theta_2(n) = \begin{cases} 1 & \text{if } \left(\frac{r_0}{d_0} > r_1 + \rho_1 + \frac{1}{2}\right) \wedge \left(r + \rho = r_1 + \rho_1\right) \\ (\log n)^{\frac{1}{2}} & \text{if } \left(\frac{r_0}{d_0} > r_1 + \rho_1 + \frac{1}{2}\right) \wedge \left(r + \rho > r_1 + \rho_1\right) \\ (\log n)^{\frac{r_0}{d_0}+\frac{3}{2}} & \text{if } \frac{r_0}{d_0} = r_1 + \rho_1 + \frac{1}{2} \\ (\log n)^{\frac{r_0}{d_0}-r_1-\rho_1} & \text{if } r_1 + \rho_1 < \frac{r_0}{d_0} < r_1 + \rho_1 + \frac{1}{2} \\ (\log\log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0} = r_1 + \rho_1 > 0 \\ 1 & \text{if } \left(\frac{r_0}{d_0} = r_1 + \rho_1 = 0\right) \vee \left(\frac{r_0}{d_0} < r_1 + \rho_1\right). \end{cases} \tag{24}$$

Theorem 3.1. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $d_0 \in \mathbb{N}$, $0 \le \rho, \rho_1 \le 1$, with $r + \rho \ge r_1 + \rho_1$, $\kappa, L, \sigma > 0$, and $\lambda_1 > \lambda_0 > 0$. There are constants $c_{1-6} > 0$ and $\nu_0 \in \mathbb{N}$ such that the following hold. Let H be a Hilbert space and let $\mathscr{F}$ be defined by (17–19). Then for all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$ and for all $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ with $n_l \ge \nu_0$ $(l_0 \le l \le l_1)$, $\mathscr{A}_\omega(f, u_0)$ is defined for all $(f, u_0) \in \mathscr{F}$, $\omega \in \Omega$,*

$$\begin{aligned}&\sup_{(f,u_0)\in\mathscr{F}} \|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|_{B(Q\times[a,b],H)} \\ &\le \; c_1 2^{-r_0 l_1} + c_1 n_{l_0}^{-r-\rho} + c_1 \sum_{l=l_0+1}^{l_1} 2^{-r_0 l} n_l^{-r_1-\rho_1} \quad (\omega \in \Omega),\end{aligned} \tag{25}$$

and for all l^* with $l_0 \le l^* \le l_1$

$$\begin{aligned}&\sup_{(f,u_0)\in\mathscr{F}} \left(\mathbb{E}\|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|^2_{B(Q\times[a,b],H)}\right)^{1/2} \\ &\le \; c_2 2^{-r_0 l_1} + c_2 (l_0+1)^{1/2} n_{l_0}^{-r-\rho-1/2} \\ &\quad + c_2 \sum_{l=l_0+1}^{l^*} (l+1)^{1/2} 2^{-r_0 l} n_l^{-r_1-\rho_1-1/2} + c_2 \sum_{l=l^*+1}^{l_1} 2^{-r_0 l} n_l^{-r_1-\rho_1}.\end{aligned} \tag{26}$$

Moreover, for each $n \in \mathbb{N}$ with $n \ge 2$ there is a choice of $l_0, l_1 \in \mathbb{N}_0$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ such that $l_0 \le l_1$, $n_l \ge \nu_0$ $(l_0 \le l \le l_1)$, and for all $\omega \in \Omega$ we have $\operatorname{card}(\mathscr{A}_\omega) \le c_3 n$ *and*

$$\sup_{(f,u_0)\in\mathscr{F}} \|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|_{B(Q\times[a,b],H)} \le c_4 n^{-\upsilon_1} \theta_1(n), \tag{27}$$

where

$$\upsilon_1 = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\rho-r_1-\rho_1}(r+\rho) & \textit{if } \frac{r_0}{d_0} > r_1+\rho_1 \\ \frac{r_0}{d_0} & \textit{if } \frac{r_0}{d_0} \le r_1+\rho_1 \end{cases} \tag{28}$$

and θ_1 was defined in (23).

Finally, for each $n \in \mathbb{N}$ with $n > 2$ there is a choice of $l_0, l_1 \in \mathbb{N}_0$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ such that $l_0 \le l_1$, $n_l \ge \nu_0$ $(l_0 \le l \le l_1)$, $\operatorname{card}(\mathscr{A}_\omega) \le c_5 n$ $(\omega \in \Omega)$, *and*

$$\sup_{(f,u_0)\in\mathscr{F}} \left(\mathbb{E}\|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|^2_{B(Q\times[a,b],H)}\right)^{1/2} \le c_6 n^{-\upsilon_2} \theta_2(n), \tag{29}$$

with

$$\upsilon_2 = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\rho-r_1-\rho_1}\left(r+\rho+\frac{1}{2}\right) & \textit{if } \frac{r_0}{d_0} > r_1+\rho_1+\frac{1}{2} \\ \frac{r_0}{d_0} & \textit{if } \frac{r_0}{d_0} \le r_1+\rho_1+\frac{1}{2} \end{cases} \tag{30}$$

and θ_2 is given by (24).

We use the following lemma which was proved in [4], where the complexity of parametric integration was studied in different but related smoothness classes.

Let $\beta, \beta_0, \beta_1 \in \mathbb{R}$. Given $l_0, l^*, l_1 \in \mathbb{N}_0$ with $l_0 \le l^* \le l_1$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$, we define

$$M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) = 2^{-\beta_0 d_0 l_1} + n_{l_0}^{-\beta} + \sum_{l=l_0+1}^{l_1} 2^{-\beta_0 d_0 l} n_l^{-\beta_1} \tag{31}$$

$$\begin{aligned} E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big) &= 2^{-\beta_0 d_0 l_1} + (l_0+1)^{1/2} n_{l_0}^{-\beta} + \sum_{l=l_0+1}^{l^*} (l+1)^{1/2} 2^{-\beta_0 d_0 l} n_l^{-\beta_1} \\ &\quad + \sum_{l=l^*+1}^{l_1} 2^{-\beta_0 d_0 l} n_l^{-\beta_1 + 1/2}. \end{aligned} \tag{32}$$

Lemma 3.2. *Let $\beta, \beta_0, \beta_1 \in \mathbb{R}$ with $\beta_0 \ge 0$ and $\beta \ge \beta_1 \ge 0$. Then there are constants $c_{1-3} > 0$ such that for each $n \in \mathbb{N}$ with $n \ge 2$ there is a choice of $l_0, l_1 \in \mathbb{N}_0$, $l_0 \le l_1$, and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ such that*

$$\sum_{l=l_0}^{l_1} n_l 2^{d_0 l} \le c_1 n \tag{33}$$

and

$$\begin{aligned} & M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) \\ \le \quad & c_2 n^{-\upsilon} \begin{cases} 1 & \text{if} \quad (\beta_0 \ne \beta_1) \vee (\beta_0 = \beta_1 = 0) \\ (\log n)^{\beta_0+1} & \text{if} \quad \beta_0 = \beta_1 > 0, \end{cases} \end{aligned} \tag{34}$$

where

$$\upsilon = \begin{cases} \frac{\beta_0 \beta}{\beta_0 + \beta - \beta_1} & \text{if} \quad \beta_0 > \beta_1 \\ \beta_0 & \text{if} \quad \beta_0 \le \beta_1. \end{cases} \tag{35}$$

Moreover, if $\beta_1 \ge 1/2$, then for each $n \in \mathbb{N}$ with $n > 2$ there is a choice of $l_0, l^*, l_1 \in \mathbb{N}_0$, $l_0 \le l^* \le l_1$, and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ satisfying (33) *and*

$$\begin{aligned} & E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big) \\ \le \quad & c_3 n^{-\upsilon} \begin{cases} 1 & \text{if} \quad \beta_0 > \beta_1 = \beta \\ (\log n)^{1/2} & \text{if} \quad \beta_0 > \beta_1 \quad \text{and} \quad \beta > \beta_1 \\ (\log n)^{\beta_0+3/2} & \text{if} \quad \beta_0 = \beta_1 \\ (\log n)^{\beta_0-\beta_1+1/2} & \text{if} \quad \beta_1 - 1/2 < \beta_0 < \beta_1 \\ (\log\log n)^{\beta_0+1} & \text{if} \quad \beta_0 = \beta_1 - 1/2. \end{cases} \end{aligned} \tag{36}$$

Proof of Theorem 3.1. First we show (25) and (26). Let $\delta_0 = (\lambda_1 - \lambda_0)/4 > 0$ and let ψ be a C^∞ function on $[0, +\infty)$ with

$$\begin{array}{ll} \psi(\tau) = 1 & \text{if} \quad 0 \le \tau \le (\lambda_0 + 2\delta_0)^2 \\ \psi(\tau) = 0 & \text{if} \quad \tau \ge (\lambda_0 + 3\delta_0)^2. \end{array}$$

For

$$f \in \mathscr{C}^{0,r,\rho}_{\mathrm{Lip}}(Q \times [a,b] \times \lambda_1 B^0_H, H; \kappa, L) \cap \mathscr{C}^{r_0,r_1,\rho_1}_{\mathrm{Lip}}(Q \times [a,b] \times \lambda_1 B^0_H, H; \kappa, L) \quad (37)$$

we put

$$\tilde{f}(s,t,x) = \begin{cases} f(s,t,x)\psi(\|x\|^2) & \text{if} \quad \|x\| < \lambda_1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\tilde{f}(s,t,x) = f(s,t,x) \quad (\|x\| \le \lambda_0 + 2\delta_0). \quad (38)$$

Moreover, due to the (infinite) differentiability of the scalar product $(x, x) = \|x\|^2$ there are $\kappa_1, L_1 > 0$ (not depending on H) such that for all f satisfying (37)

$$\tilde{f} \in \mathscr{C}^{0,r,\rho}_{\mathrm{Lip}}(Q \times [a,b] \times H, H; \kappa_1, L_1) \cap \mathscr{C}^{r_0,r_1,\rho_1}_{\mathrm{Lip}}(Q \times [a,b] \times H, H; \kappa_1, L_1). \quad (39)$$

Let $u_0 \in \sigma B_{C^r(Q,H)}$ and assume that $(f, u_0) \in \mathscr{F}$. Then, by assumption, the solution $u(s,t)$ of (4–5) exists and fulfills

$$\sup_{s \in Q, t \in [a,b]} \|u(s,t)\| \le \lambda_0. \quad (40)$$

Consequently,

$$\frac{d}{dt} u(s,t) = f(s,t,u(s,t)) = \tilde{f}(s,t,u(s,t)) \quad (s \in Q,\ t \in [a,b]),$$

which implies

$$\mathscr{S}(\tilde{f}, u_0) = \mathscr{S}(f, u_0). \quad (41)$$

Let us denote $h_l = (b-a)/n_l$ and

$$r(l) = \begin{cases} r & \text{if} \quad l = l_0 \\ r_1 & \text{if} \quad l_0 < l \le l_1. \end{cases} \quad (42)$$

Now we show that for $(f, u_0) \in \mathscr{F}$ and $\omega \in \Omega$, algorithm $\mathscr{A}_\omega$ is defined and

$$\mathscr{A}_\omega(\tilde{f}, u_0) = \mathscr{A}_\omega(f, u_0). \quad (43)$$

First we consider the case $r + \rho > 0$. It follows from (39) and Theorem 3.2 of [7] that there is a $\nu_1 > 0$ such that for all $l_0 \le l \le l_1$, $n_l \ge \nu_1$, $\omega \in \Omega$, $s \in Q$

$$\|\mathscr{S}(\tilde{f}_s, u_0(s)) - A^{r(l)}_{n_l,\omega}(\tilde{f}_s, u_0(s))\|_{B([a,b],H)} \le \delta_0,$$

hence, by (40) and (41),

$$\|A^{r(l)}_{n_l,\omega}(\tilde f_s, u_0(s))\|_{B([a,b],H)} \le \lambda_0 + \delta_0. \tag{44}$$

Now we fix l with $l_0 \le l \le l_1$, $n_l \ge \nu_1$, $\omega \in \Omega$, $s \in Q$. Let $\tilde u_k(s)$ $(0 \le k \le n_l)$, $\tilde p_{k,j}(s,\,\cdot\,)$, and $\tilde q_{k,j}(s)$ $(0 \le k \le n_l - 1,\ 0 \le j \le r(l))$ be the sequences arising in the definition (6–12) of $A^{r(l)}_{n_l,\omega}(\tilde f_s, u_0(s))$, and let $u_k(s)$, $p_{k,j}(s,\,\cdot\,)$, and $q_{k,j}(s)$ be the corresponding sequences for $(f_s, u_0(s))$, as far as they are defined on $[a,b]\times\lambda_1 B^0_H$ (see (6), (8), and (11)). By (44), for $t \in [t_k, t_{k+1}]$,

$$\|\tilde p_{k,r(l)}(s,t)\| \le \lambda_0 + \delta_0$$

and therefore also

$$\|\tilde u_k(s)\| \le \lambda_0 + \delta_0. \tag{45}$$

By (7) and (9–10), for $0 \le j \le r(l)$

$$\|\tilde p_{k,j}(s,t) - \tilde u_k(s)\| \le c_0(r(l))\kappa_1 h_l,$$

where

$$c_0(0) = 1, \quad c_0(m) = \max_{1\le j\le m} \|P_{k,j}\|_{\mathscr{L}(C([t_k,t_{k+1}],H))} \quad (m \ge 1)$$

and $P_{k,j}$ is the Lagrange interpolation operator introduced in Section 2. Note that $c_0(m)$ is a constant depending only on m. Together with (45) this yields

$$\begin{aligned}\|\tilde p_{k,j}(s,t)\| &\le \lambda_0 + \delta_0 + c_0(r(l))\kappa_1 h_l \\ &\le \lambda_0 + 2\delta_0 \qquad (t \in [t_k, t_{k+1}],\ 0 \le j \le r(l)),\end{aligned} \tag{46}$$

provided $n_l \ge \nu_0$, with a suitably chosen $\nu_0 \ge \nu_1$.

We prove that for $0 \le k \le n_l$ the following holds:

$$u_k(s) \text{ is defined and } u_k(s) = \tilde u_k(s), \tag{47}$$

and, if $k \le n_l - 1$, then for all j with $0 \le j \le r(l)$

$$p_{k,j}(s,\,\cdot\,) \text{ is defined and } p_{k,j}(s,\,\cdot\,) = \tilde p_{k,j}(s,\,\cdot\,). \tag{48}$$

First we show that (47) implies (48). Suppose (47) holds for some $0 \le k \le n-1$. We argue by induction over j. Let $j = 0$. By (38), (45), and (47),

$$f(s, t_k, u_k(s)) = f(s, t_k, \tilde u_k(s)) = \tilde f(s, t_k, \tilde u_k(s)),$$

$p_{k,0}(s,\,\cdot\,)$ is defined, and

$$\begin{aligned}p_{k,0}(s,t) &= u_k(s) + f(s,t_k,u_k(s))(t - t_k) \\ &= u_k(s) + \tilde f(s,t_k,\tilde u_k(s))(t-t_k) = \tilde p_{k,0}(s,t).\end{aligned}$$

This is (48) for $j = 0$. Next suppose (48) holds for some j with $0 \le j < r(l)$. Then

$$p_{k,j}(s, t_{k,j+1,i}) = \tilde{p}_{k,j}(s, t_{k,j+1,i}) \quad (i = 0, \dots, j+1), \tag{49}$$

and therefore, by (46),

$$\|p_{k,j}(s, t_{k,j+1,i})\| \le \lambda_0 + 2\delta_0. \tag{50}$$

It follows that $p_{k,j+1}(s, \cdot\,)$ is defined. Using (38), (49), and (50), we get

$$f(s, t_{k,j+1,i}, p_{k,j}(s, t_{k,j+1,i})) = \tilde{f}(s, t_{k,j+1,i}, \tilde{p}_{k,j}(s, t_{k,j+1,i})),$$

therefore we also have $q_{k,j}(s) = \tilde{q}_{k,j}(s)$ and

$$\begin{aligned} p_{k,j+1}(s,t) &= u_k(s) + \int_{t_k}^{t} (P_{k,j+1} q_{k,j}(s))(\tau) d\tau \\ &= \tilde{u}_k(s) + \int_{t_k}^{t} (P_{k,j+1} \tilde{q}_{k,j}(s))(\tau) d\tau = \tilde{p}_{k,j+1}(s,t). \end{aligned}$$

This completes the induction over j and the proof that (47) implies (48).

It remains to show (47). We use induction over k. The case $k = 0$ holds by definition. Now we assume that (47) and therefore also (48) hold for some k with $0 \le k \le n - 1$. From (46) and (48) we conclude

$$\|p_{k,r(l)}(s,t)\| = \|\tilde{p}_{k,r(l)}(s,t)\| \le \lambda_0 + 2\delta_0 \quad (t \in [t_k, t_{k+1}]),$$

which shows that $u_{k+1}(s)$ is defined and

$$\begin{aligned} u_{k+1}(s) &= p_{k,r(l)}(s, t_{k+1}) + h_l\Big(f(s, \xi_{k+1}, p_{k,r(l)}(s, \xi_{k+1})) - \frac{\partial p_{k,r(l)}}{\partial t}(s, \xi_{k+1}) \Big) \\ &= \tilde{p}_{k,r(l)}(s, t_{k+1}) + h_l\Big(\tilde{f}(s, \xi_{k+1}, \tilde{p}_{k,r(l)}(s, \xi_{k+1})) - \frac{\partial \tilde{p}_{k,r(l)}}{\partial t}(s, \xi_{k+1}) \Big) \\ &= \tilde{u}_{k+1}(s). \end{aligned}$$

This gives (47) for $k+1$, completes the induction over k and the proof of (47–48). It follows that $A^{r(l)}_{n_l,\omega}(f_s, u_0(s))$ is defined and

$$A^{r(l)}_{n_l,\omega}(\tilde{f}_s, u_0(s)) = A^{r(l)}_{n_l,\omega}(f_s, u_0(s)).$$

Consequently, $\mathscr{A}_\omega(f, u_0)$ is defined and (43) holds for $r + \rho > 0$.

In the case $r + \rho = 0$ we have, by assumption, also $r_1 = \rho_1 = 0$ and therefore, by (42), $r(l) = 0$ $(l_0 \le l \le l_1)$. By definition of $\mathscr{F}$, $A^0_{n_l,\omega}(f_s, u_0(s))$ is defined for $l_0 \le l \le l_1$ and $s \in Q$, so $\mathscr{A}_\omega(f, u_0)$ is defined. Fix l with $l_0 \le l \le l_1$, $n_l \in \mathbb{N}$, $\omega \in \Omega$, $s \in Q$. Let $\tilde{u}_k(s)$ and $\tilde{p}_{k,0}(s, \cdot\,)$ $(0 \le k \le n_l - 1)$ be the resulting sequences from $A^0_{n_l,\omega}(\tilde{f}_s, u_0(s))$, and $u_k(s)$, $p_{k,0}(s, \cdot\,)$ the respective sequences from $A^0_{n_l,\omega}(f_s, u_0(s))$. Then (19) implies

$$\|u_k(s)\| \le \lambda_0 \quad (0 \le k \le n_l) \tag{51}$$

$$\|p_{k,0}(s,t)\| \le \lambda_0 \quad (t \in [t_k, t_{k+1}],\ 0 \le k \le n_l - 1). \tag{52}$$

For $0 \le k \le n_l$ the following holds:

$$u_k(s) = \tilde{u}_k(s) \tag{53}$$
$$p_{k,0}(s,\,\cdot\,) = \tilde{p}_{k,0}(s,\,\cdot\,) \quad (k \le n_l - 1). \tag{54}$$

This follows readily by induction as above. Indeed, the case $k = 0$ of (53) is clear, and if (53) holds for some k, we get, using (15–16), (38), and (51–52),

$$\begin{aligned} p_{k,0}(s,t) &= u_k(s) + f(s,t_k,u_k(s))(t-t_k) \\ &= \tilde{u}_k(s) + \tilde{f}(s,t_k,\tilde{u}_k(s))(t-t_k) = \tilde{p}_{k,0}(s,t) \end{aligned}$$

and

$$\begin{aligned} u_{k+1}(s) &= u_k(s) + h_l f(s,\xi_{k+1},p_{k,0}(s,\xi_{k+1})) \\ &= \tilde{u}_k(s) + h_l \tilde{f}(s,\xi_{k+1},\tilde{p}_{k,0}(s,\xi_{k+1})) = \tilde{u}_{k+1}(s). \end{aligned}$$

This shows

$$A^0_{n_l,\omega}(\tilde{f}_s, u_0(s)) = A^0_{n_l,\omega}(f_s, u_0(s))$$

and consequently (43) for $r + \rho = 0$. Now the proof of (25) and (26) is finished by combining (39), (41), (43), and Theorem 4.1 of [3].

Next we derive (27) and (29) from (25), (26), and Lemma 3.2. To show (27) we define

$$\beta = r + \rho, \quad \beta_0 = \frac{r_0}{d_0}, \quad \beta_1 = r_1 + \rho_1, \tag{55}$$

which together with (28) and (35) gives $\upsilon = \upsilon_1$. Furthermore, (25) and (31) yield

$$\sup_{(f,u_0)\in\mathscr{F}} \|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|_{B(Q\times[a,b],H)} \le c\, M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) \tag{56}$$

for all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ with $n_l \ge \nu_0$ $(l_0 \le l \le l_1)$. Now (27) follows from (22), (55–56), and (34) of Lemma 3.2. Finally, to check (29), we set

$$\beta = r + \rho + \frac{1}{2}, \quad \beta_0 = \frac{r_0}{d_0}, \quad \beta_1 = r_1 + \rho_1 + \frac{1}{2}, \tag{57}$$

which by (30) and (35) implies $\upsilon = \upsilon_2$. We infer from (26) and (32) that

$$\sup_{(f,u_0)\in\mathscr{F}} \big(\mathbb{E}\|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|^2_{B(Q\times[a,b],H)}\big)^{1/2} \le c\, E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big) \tag{58}$$

for all $l_0, l^*, l_1 \in \mathbb{N}_0$ with $l_0 \le l^* \le l_1$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ with $n_l \ge \nu_0$ $(l_0 \le l \le l_1)$. Now (29) in the first five cases of (24) is a consequence of (22), (57–58), and (36) of Lemma 3.2. In the last case of (24) relation (29) follows directly from (23) and (27).

□

Remark 3.3. The case $r + \rho < r_1 + \rho_1$ is left out in Theorem 3.1 because it represents nothing new – it is (up to constants) the same as the case $r = r_1$, $\rho = \rho_1$. This is easily seen directly, we also refer to the argument in Remark 4.2 of [3], which carries over to our situation.

4 Complexity

For the framework of information-based complexity theory we refer to [10, 9], details on the notions used here can be found in [5, 6]. In the terminology of [5, 6], the parametric initial value problem is given by the tuple

$$(\mathscr{S}, \mathscr{F}, B(Q \times [a,b], H), H, \Lambda).$$

The first three components describe the already defined in (20) solution operator $\mathscr{S} : \mathscr{F} \to B(Q \times [a,b], H)$. Information to be used about $(f, u_0) \in \mathscr{F}$ takes values in H (the fourth component) and the set Λ of information functionals is of the form

$$\Lambda \quad = \quad \{\delta_{s,t,z} : \ s \in Q, t \in [a,b], z \in \lambda_1 B_H^0\} \cup \{\delta_s : \ s \in Q\}, \tag{59}$$

where $\delta_{s,t,z}(f, u_0) = f(s,t,z)$ and $\delta_s(f, u_0) = u_0(s)$. Thus, information is standard, that means, consists of values of f and u_0. The n-th minimal error of $\mathscr{S}$ on $\mathscr{F}$ in the deterministic, respectively randomized setting, that is, the minimal possible error among all deterministic, respectively randomized algorithms that use at most n information functionals, is denoted by $e_n^{\rm det}(\mathscr{S}, \mathscr{F})$, respectively $e_n^{\rm ran}(\mathscr{S}, \mathscr{F})$.

Given two sequences of nonnegative reals $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, we write $a_n \preceq b_n$ if there is a constant $c > 0$ and an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $a_n \le c b_n$. The notation $a_n \asymp b_n$ means $a_n \preceq b_n$ and $b_n \preceq a_n$.

Next we state the main result of this paper, which settles the complexity of the parametric initial value problem. It gives the sharp order of the deterministic and randomized minimal errors, except for some limit cases, where logarithmic gaps remain. Together with Theorem 3.1, it also shows the optimality of the multilevel algorithm (21).

Theorem 4.1. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $d_0 \in \mathbb{N}$, $0 \le \rho, \rho_1 \le 1$, with $r + \rho \ge r_1 + \rho_1$, $\kappa, L, \sigma > 0$, and $\lambda_1 > \lambda_0 > 0$. Let H be a Hilbert space, and let $\mathscr{F}$ be defined by* (17–19)*. Then in the deterministic setting,*

$$e_n^{\rm det}(\mathscr{S}, \mathscr{F}) \asymp n^{-\upsilon_1} \quad \text{if} \quad \left(\tfrac{r_0}{d_0} \ne r_1 + \rho_1\right) \vee \left(\tfrac{r_0}{d_0} = r_1 + \rho_1 = 0\right)$$

and

$$n^{-\upsilon_1} \preceq e_n^{\rm det}(\mathscr{S}, \mathscr{F}) \preceq n^{-\upsilon_1} (\log n)^{\frac{r_0}{d_0}+1} \quad \text{if} \quad \tfrac{r_0}{d_0} = r_1 + \rho_1 > 0,$$

where υ_1 is given by (28)*. In the randomized setting,*

$$e_n^{\rm ran}(\mathscr{S}, \mathscr{F}) \asymp n^{-\upsilon_2}\theta_2(n) \quad \text{if} \quad \left(\left(\tfrac{r_0}{d_0} \ne r_1 + \rho_1 + \tfrac12\right) \wedge \left(\tfrac{r_0}{d_0} \ne r_1 + \rho_1\right)\right) \vee \left(\tfrac{r_0}{d_0} = r_1 + \rho_1 = 0\right)$$

and, in the limit cases,

$$\begin{aligned} n^{-\upsilon_2}(\log n)^{\frac{1}{2}} &\preceq e_n^{\rm ran}(\mathscr{S},\mathscr{F}) \preceq n^{-\upsilon_2}(\log n)^{\frac{r_0}{d_0}+\frac{3}{2}} \quad && \textit{if} \quad \tfrac{r_0}{d_0} = r_1+\rho_1+\tfrac{1}{2} \\ n^{-\upsilon_2} &\preceq e_n^{\rm ran}(\mathscr{S},\mathscr{F}) \preceq n^{-\upsilon_2}(\log\log n)^{\frac{r_0}{d_0}+1} \quad && \textit{if} \quad \tfrac{r_0}{d_0} = r_1+\rho_1 > 0, \end{aligned}$$

where υ_2 *is defined in* (30) *and* θ_2 *in* (24).

To prove the lower bounds we will reduce the parametric initial value problem to definite parametric integration. We consider the space $C^{r_0,r,\rho}(Q\times[a,b],H)$ of continuous functions $f: Q\times[a,b]\to H$ having for $\alpha=(\alpha_0,\alpha_1)\in\mathbb{N}_0^2$ with $\alpha_0\le r_0$, $\alpha_1\le r$ continuous partial derivatives $\frac{\partial^{|\alpha|}f(s,t)}{\partial s^{\alpha_0}\partial t^{\alpha_1}}$ with the following property: there is a constant $c\ge 0$ such that for $\alpha_0\le r_0$, $\alpha_1\le r$

$$\left\|\frac{\partial^{|\alpha|}f(s,t)}{\partial s^{\alpha_0}\partial t^{\alpha_1}}\right\| \le c \quad (s\in Q, t\in[a,b]) \tag{60}$$

and

$$\left\|\frac{\partial^{|\alpha|}f(s,t_1)}{\partial s^{\alpha_0}\partial t^{\alpha_1}} - \frac{\partial^{|\alpha|}f(s,t_2)}{\partial s^{\alpha_0}\partial t^{\alpha_1}}\right\| \le c|t_1-t_2|^\rho \quad (s\in Q, t_1,t_2\in[a,b]). \tag{61}$$

The norm $\|f\|_{C^{r_0,r,\rho}(Q\times[a,b],H)}$ is defined to be the smallest constant $c\ge 0$ satisfying (60–61). If $H=\mathbb{R}$, we write $C^{r_0,r,\rho}(Q\times[a,b])$. Considering functions on $Q\times[a,b]$ as functions on $Q\times[a,b]\times\lambda_1 B_H^0$ not depending on $z\in\lambda_1 B_H^0$ and comparing (1–3) with (60–61), we see that for all $L>0$

$$\mathscr{C}_{\rm Lip}^{r_0,r,\rho}(Q\times[a,b]\times\lambda_1 B_H^0, H;\kappa,L)\cap C(Q\times[a,b],H) = \kappa B_{C^{r_0,r,\rho}(Q\times[a,b],H)}. \tag{62}$$

The operator of definite parametric integration $\mathscr{S}_0 : C(Q\times[a,b])\to C(Q)$ is given for $f\in C(Q\times[a,b])$ by

$$(\mathscr{S}_0 f)(s) = \int_a^b f(s,t)dt \quad (s\in Q). \tag{63}$$

In connection with $\mathscr{S}_0$ we consider the following class of information functionals

$$\Lambda_0 = \{\delta_{s,t} : s\in Q, t\in[a,b]\}, \quad \delta_{s,t}(f) = f(s,t). \tag{64}$$

Now let ϕ_0 be a C^∞ function on $\mathbb{R}^{d_0}$ with support in Q and

$$\phi_0\left(\frac{1}{2},\dots,\frac{1}{2}\right) = 1. \tag{65}$$

Furthermore, let ϕ_1 be a C^∞ function on $\mathbb{R}$ with support in $[a,b]$ and $\int_a^b \phi_1(t)dt \ne 0$. Let $m_0,m_1\in\mathbb{N}$, let Q_i $(i=1,\dots,m_0^{d_0})$ be the subdivision of Q into $m_0^{d_0}$ cubes of disjoint interior of sidelength m_0^{-1}. Let s_i be the point in Q_i with minimal coordinates. For $j = 0, \ldots, m_1 - 1$ put $t_j = a + j(b-a)/m_1$ and define for $s \in Q$, $t \in [a, b]$, $i = 1, \ldots, m_0^{d_0}$

$$\phi_{0,i}(s) = \phi_0(m_0(s - s_i)), \quad \phi_{1,j}(t) = \phi_1(a + m_1(t - t_j))$$

and

$$\psi_{ij}(s,t) = \phi_{0,i}(s)\phi_{1,j}(t).$$

Denote

$$\Psi^0_{m_0,m_1} = \left\{ \sum_{i=1}^{m_0^{d_0}} \sum_{j=0}^{m_1-1} \delta_{ij}\psi_{ij} : \delta_{ij} \in [-1,1],\, i = 1, \ldots, m_0^{d_0}, j = 0, \ldots, m_1 - 1 \right\}.$$

We use another technical estimate from [4], which, in turn, is based on lower estimates obtained in [8]. For $\gamma, \gamma_0, \gamma_1 \in \mathbb{R}$ we define

$$\Psi^{\gamma,\gamma_0,\gamma_1}_{m_0,m_1} = \min\left(m_1^{-\gamma}, m_0^{-\gamma_0} m_1^{-\gamma_1}\right) \Psi^0_{m_0,m_1}. \tag{66}$$

We recall Lemma 4.5 of [4], with $d = 1$, which corresponds to our situation.

Lemma 4.2. *Let $\gamma, \gamma_0, \gamma_1 \in \mathbb{R}$ with $\gamma_0 \geq 0$ and $\gamma \geq \gamma_1 \geq 0$. Then there are constants $c_1, c_2 > 0$ such that for each $n \in \mathbb{N}$ with $n \geq 2$ there is a choice of $m_0, m_1 \in \mathbb{N}_0$ fulfilling*

$$m_0^{d_0} m_1 \geq 4n \tag{67}$$

and

$$e_n^{\det}(\mathscr{S}_0, \Psi^{\gamma,\gamma_0,\gamma_1}_{m_0,m_1}) \;\geq\; c_1 n^{-\upsilon_3},$$

where υ_3 is defined by

$$\upsilon_3 \;=\; \begin{cases} \frac{\gamma_0\gamma}{\gamma_0 + (\gamma-\gamma_1)d_0} & \text{if} \quad \gamma_0/d_0 > \gamma_1 \\ \frac{\gamma_0}{d_0} & \text{if} \quad \gamma_0/d_0 \leq \gamma_1. \end{cases} \tag{68}$$

Furthermore, for each $n \in \mathbb{N}$ with $n > 2$ there is a choice of $m_0, m_1 \in \mathbb{N}_0$ such that (67) holds and

$$\begin{aligned} & e_n^{\mathrm{ran}}(\mathscr{S}_0, \Psi^{\gamma,\gamma_0,\gamma_1}_{m_0,m_1}) \\ & \geq \;\; c_2 n^{-\upsilon_4} \begin{cases} 1 & \text{if} \quad (\gamma_0/d_0 > \gamma_1 + 1/2) \wedge (\gamma = \gamma_1) \\ (\log n)^{1/2} & \text{if} \quad (\gamma_0/d_0 > \gamma_1 + 1/2) \wedge (\gamma > \gamma_1) \\ (\log n)^{\gamma_0/d_0 - \gamma_1} & \text{if} \quad \gamma_1 < \gamma_0/d_0 \leq \gamma_1 + 1/2 \\ 1 & \text{if} \quad \gamma_0/d_0 \leq \gamma_1, \end{cases} \end{aligned}$$

with

$$\upsilon_4 \;=\; \begin{cases} \frac{\gamma_0(\gamma+1/2)}{\gamma_0 + (\gamma-\gamma_1)d_0} & \text{if} \quad \gamma_0/d_0 > \gamma_1 + 1/2 \\ \frac{\gamma_0}{d_0} & \text{if} \quad \gamma_0/d_0 \leq \gamma_1 + 1/2. \end{cases} \tag{69}$$

Proof of Theorem 4.1. The upper bounds follow from Theorem 3.1. To show the lower bounds, let $z_0 \in H$ with $\|z_0\| = 1$ and define

$$V_1 : \kappa B_{C(Q\times[a,b])} \to \mathscr{C}^{0,0,0}_{\mathrm{Lip}}(Q \times [a,b] \times \lambda_1 B^0_H, H; \kappa, L) \times \sigma B_{C(Q)} \tag{70}$$

for $f \in \kappa B_{C(Q\times[a,b])}$ by

$$V_1 f = (\tilde{f}, 0), \tag{71}$$

where

$$\tilde{f}(s,t,z) = f(s,t)z_0 \quad (s \in Q,\, t \in [a,b],\, z \in \lambda_1 B^0_H). \tag{72}$$

By (59) and (64), information on $V_1 f$ transforms into information on f as follows

$$\delta_{s,t,z}(V_1 f) = \tilde{f}(s,t,z) = f(s,t)z_0 = \delta_{s,t}(f)z_0 \tag{73}$$

$$\delta_s(V_1 f) = 0. \tag{74}$$

Furthermore, let $V_2 : B(Q \times [a,b], H) \to B(Q)$ be given for $g \in B(Q \times [a,b], H)$ by

$$(V_2 g)(s) = (g(s,b), z_0) \quad (s \in Q). \tag{75}$$

Clearly,

$$\|V_2\| = 1. \tag{76}$$

Let

$$\eta_i = s_i + \left(\frac{1}{2m_0}, \dots, \frac{1}{2m_0}\right)$$

be the center of Q_i and let $T_{m_0} : B(Q) \to C(Q)$ be given for $v \in B(Q)$ by

$$T_{m_0} v = \sum_{i=1}^{m_0^{d_0}} v(\eta_i)\phi_{0,i}.$$

There is a constant $c_1 > 0$ such that for all $m_0 \in \mathbb{N}$

$$\|T_{m_0}\| \le c_1 \tag{77}$$

and, because of (65), we have

$$T_{m_0} v = v \qquad \big(v \in \operatorname{span}\{\phi_{0,i} : i = 1, \dots, m_0^{d_0}\}\big). \tag{78}$$

Moreover, the solution $u = \mathscr{S}(\tilde{f}, 0)$ of

$$\frac{d}{dt}u(s,t) = \tilde{f}(s,t,u(s,t)) = f(s,t)z_0 \qquad (s \in Q, t \in [a,b])$$

$$u(s,a) = 0 \quad (s \in Q)$$

is

$$u(s,t) = \int_a^t f(s,\tau)d\tau\, z_0. \tag{79}$$

It follows from (63), (71), (75), (78), and (79) that

$$\mathscr{S}_0 f = T_{m_0} V_2 \mathscr{S}(V_1 f) \qquad \big(f \in \operatorname{span} \Psi^0_{m_0,m_1}\big). \tag{80}$$

By (79), $\|f\|_{C(Q\times[a,b])} \le (b-a)^{-1}\lambda_0$, implies that

$$\sup_{s\in Q, t\in[a,b]} \|u(s,t)\| \le \lambda_0 \tag{81}$$

and, using (15) and (16), for all $n \in \mathbb{N}$, $\omega \in \Omega$, $s \in Q$, $A^0_{n,\omega}(\tilde f_s, 0)$ is defined on $[a,b] \times \lambda_1 B^0_H$ and

$$\sup_{s\in Q} \left\| A^0_{n,\omega}(\tilde f_s, 0)\right\|_{B([a,b],H)} \le \lambda_0. \tag{82}$$

It is readily checked that there is a constant $c_2 > 0$ such that for all $m_0, m_1 \in \mathbb{N}$, $\psi \in \Psi^0_{m_0,m_1}$

$$\begin{aligned} \|\psi\|_{C^{0,r,\rho}(Q\times[a,b])} &\le c_2 m_1^{r+\rho} \\ \|\psi\|_{C^{r_0,r_1,\rho_1}(Q\times[a,b])} &\le c_2 m_0^{r_0} m_1^{r_1+\rho_1}. \end{aligned}$$

Setting $c_3 = c_2^{-1} \min\left(\kappa, (b-a)^{-1}\lambda_0\right)$, it follows that for all $m_0, m_1 \in \mathbb{N}$

$$c_3 \Psi^{r+\rho, r_0, r_1+\rho_1}_{m_0,m_1} \subseteq \min\big(\kappa, (b-a)^{-1}\lambda_0\big)\left(B_{C^{0,r,\rho}(Q\times[a,b])} \cap B_{C^{r_0,r_1,\rho_1}(Q\times[a,b])}\right)$$

and therefore, taking into account (62), (81), and (82)

$$V_1\left(c_3 \Psi^{r+\rho,r_0,r_1+\rho_1}_{m_0,m_1}\right) \subseteq \mathscr{F}. \tag{83}$$

It follows from (73–74), (80), and (83) that the problem

$$(\mathscr{S}_0, c_3\Psi^{r+\rho,r_0,r_1+\rho_1}_{m_0,m_1}, C(Q), \mathbb{R}, \Lambda_0)$$

reduces to

$$(\mathscr{S}, \mathscr{F}, B(Q\times[a,b],H), H, \Lambda)$$

(see Section 3 of [6]). Consequently, by (76) and (77), for all $n, m_0, m_1 \in \mathbb{N}$

$$e_n^{\rm set}(\mathscr{S},\mathscr{F}) \ge c_1^{-1} e_n^{\rm set}(\mathscr{S}_0, c_3\Psi^{r+\rho,r_0,r_1+\rho_1}_{m_0,m_1}) = c_1^{-1}c_3\, e_n^{\rm set}(\mathscr{S}_0, \Psi^{r+\rho,r_0,r_1+\rho_1}_{m_0,m_1}), \tag{84}$$

where set $\in \{\text{det}, \text{ran}\}$.

Now we let $\gamma = r+\rho$, $\gamma_0 = r_0$, $\gamma_1 = r_1 + \rho_1$ and get from (28), (30), (68), and (69) that $\upsilon_3 = \upsilon_1$ and $\upsilon_4 = \upsilon_2$. This together with (84) and Lemma 4.2 proves the lower bounds of Theorem 4.1.

□

As an example let us just mention the special case of functions with dominating mixed smoothness. For further discussion of special classes we refer to

Section 6 of [3]. Based on the results above, this discussion is easily extended to the present context. If $r = r_1$, $\rho = \rho_1$, then $\mathscr{F}$ is the set of all

$$(f, u_0) \quad \in \quad \mathscr{C}^{r_0,r,\rho}_{\mathrm{Lip}}(Q \times [a,b] \times \lambda_1 B^0_H, H; \kappa, L) \times \sigma B_{C^{r_0}(Q,H)}$$

which fulfill (18) and (19). We neglect logarithmic factors and write $a_n \asymp_{\log} b_n$ iff there are constants $c_1, c_2 > 0$, $n_0 \in \mathbb{N}$, and $\theta_1, \theta_2 \in \mathbb{R}$ such that for all $n \geq n_0$, $c_1 a_n (\log(n+1))^{\theta_1} \leq b_n \leq c_2 a_n (\log(n+1))^{\theta_2}$. From Theorem 4.1 we obtain

Corollary 4.3. *Let $r_0, r \in \mathbb{N}_0$, $0 \leq \rho \leq 1$, $r = r_1$, $\rho = \rho_1$. Then*

$$e^{\mathrm{det}}_n(\mathscr{S}, \mathscr{F}) \quad \asymp_{\log} \quad n^{-\min\left(r+\rho,\, \frac{r_0}{d_0}\right)}$$

$$e^{\mathrm{ran}}_n(\mathscr{S}, \mathscr{F}) \quad \asymp_{\log} \quad n^{-\min\left(r+\rho+\frac{1}{2},\, \frac{r_0}{d_0}\right)}.$$

It follows that for $r_0/d_0 \geq r+\rho+\frac{1}{2}$ the best Monte Carlo methods are superior to the best deterministic ones by an order of $n^{-1/2}$. If $r_0/d_0 < r + \rho + \frac{1}{2}$, the gain decreases, until for $r_0/d_0 \leq r + \rho$ the optimal rates of deterministic and randomized methods become the same.

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