On the complexity of parametric ODEs and related problems

Stefan Heinrich
Department of Computer Science
University of Kaiserslautern
D-67653 Kaiserslautern, Germany
e-mail: heinrich@informatik.uni-kl.de

Abstract

We present an iterative Monte Carlo procedure to solve initial value problems for systems of ordinary differential equations depending on a parameter. It is based on a multilevel Monte Carlo algorithm for parametric indefinite integration. As an application, we also obtain a respective method for solving almost linear first order partial differential equations. We also consider deterministic algorithms.

We study the convergence and, in the framework of information-based complexity, the minimal errors and show that the developed algorithms are of optimal order (in some limit cases up to logarithmic factors). This way we extend recent complexity results on parametric ordinary differential equations. Moreover, we obtain the complexity of almost linear first-order partial differential equations, which has not been analyzed before.

1 Introduction

Monte Carlo (one-level) methods for integrals depending on a parameter were first considered in [8]. Multilevel Monte Carlo methods for parametric integration were developed in [14], where the problem was studied for the first time in the frame of information-based complexity theory (IBC). These investigations were continued in [3], where the complexity of parametric indefinite integration was studied for the first time.

Recently there arose considerable interest in the numerical solution of various parametric problems, also in connection with random partial differential equations, see [1, 7, 18, 19] and references therein. Deterministic methods for solving parametric initial value problems for systems of ordinary differential equations (ODEs) were first considered in [9].

The study of ODEs in IBC was begun in [15] for the deterministic case and in [16, 17] for the stochastic case. The complexity in the randomized setting was further studied in [13, 2, 12]. The complexity of parametric initial value problems for systems of ODEs was investigated in [4] and [5], where multilevel Monte Carlo algorithms for this problem were developed and shown to be of optimal order.

Here we study function classes satisfying a weaker Lipschitz condition than those considered in [4]. This is needed for the applications to the complexity analysis for almost linear partial differential equations (PDEs). Moreover, we present a new approach to solve initial value problems for ODEs depending on a parameter. We develop an iterative Monte Carlo procedure, based on a multilevel algorithm for parametric indefinite integration. This leads to an iterative multilevel Monte Carlo method for solving almost linear first order PDEs. We also consider deterministic algorithms.

We prove convergence rates, determine the minimal errors in the framework of IBC, and show that the developed algorithms are of optimal order (in some limit cases up to logarithmic factors). This way we extend recent complexity results of [4] on parametric ordinary differential equations. Moreover, the complexity of almost linear first-order partial differential equation is determined, a topic, which has not been considered before.

The paper is organized as follows. In Section 2 we provide the needed notation. In Sections 3 and 4 we recall the algorithms from [3] on Banach space valued and parametric indefinite integration, respectively, and improve some convergence results. Section 5 contains the main results. Based on the results of Section 4 we study the iterative solution of initial value problems for parametric ordinary differential equations. Finally, in Section 6 we apply the results of Section 5 to the analysis of the complexity of almost linear partial differential equations.

2 Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. For a Banach space X the norm is denoted by $\| \|_X$, the closed unit ball by B_X, the identity mapping on X by I_X, and the dual space by X^*. The Euclidean norm on $\mathbb{R}^d$ ($d \in \mathbb{N}$) is denoted by $\| \|_{\mathbb{R}^d}$. Given another Banach space Y, we let $\mathscr{L}(X, Y)$ be the space of bounded linear mappings $T : X \to Y$ endowed with the canonical norm. If $X = Y$, we write $\mathscr{L}(X)$ instead of $\mathscr{L}(X, X)$. We assume all considered Banach spaces to be defined over the field of reals $\mathbb{R}$.

Concerning constants, we make the convention that the same symbol c, c_1, $c_2, \dots$ may denote different constants, even in a sequence of relations. Furthermore, we use the following order notation: For nonnegative reals $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we write $a_n \preceq b_n$ if there are constants $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $a_n \leq cb_n$. Finally, $a_n \asymp b_n$ stands for $a_n \preceq b_n$ and $b_n \preceq a_n$. If not specified, the function log always means $\log_2$.

Given a set $D \subseteq \mathbb{R}^d$ which is the closure of an open set, and a Banach space X, we define $C^r(D, X)$ to be the space of all functions $f : D \to X$ which are r-times continuously differentiable in the interiour of D and which together with their derivatives up to order r are bounded and possess continuous extensions to all of D. This space is equipped with the norm

$$\|f\|_{C^r(D,X)} = \sup_{|\alpha|\le r,\, s\in D} \left\| \frac{\partial^{|\alpha|} f(s)}{\partial s^\alpha} \right\|_X$$

with $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $\alpha = |\alpha_1| + \dots + |\alpha_d|$. For $r = 0$ we also write $C(D,X)$ and if $X = \mathbb{R}$, we also write $C^r(D)$ and $C(D)$.

The type 2 constant of a Banach space X is denoted by $\tau_2(X)$. We refer to [20] as well as to the introductions in [3, 4] for this notion and related facts. The injective tensor product of Banach spaces X and Y is denoted by $X \otimes_\lambda Y$. Definitions and background on tensor products can be found in [6, 21], see also the introduction to [3]. Let us mention, in particular, the canonical isometric identification

$$C(D,X) = X \otimes_\lambda C(D) \tag{1}$$

for compact $D \subset \mathbb{R}^d$. We also note that for Banach spaces X_1, X_2, Y_1, Y_2 and operators $T_1 \in \mathscr{L}(X_1, Y_1)$, $T_2 \in \mathscr{L}(X_2, Y_2)$, the algebraic tensor product $T_1 \otimes T_2 : X_1 \otimes X_2 \to Y_1 \otimes Y_2$ extends to a bounded linear operator $T_1 \otimes T_2 \in \mathscr{L}(X_1 \otimes_\lambda X_2, Y_1 \otimes_\lambda Y_2)$ with

$$\|T_1 \otimes T_2\|_{\mathscr{L}(X_1\otimes_\lambda X_2, Y_1 \otimes_\lambda Y_2)} = \|T_1\|_{\mathscr{L}(X_1,Y_1)} \|T_2\|_{\mathscr{L}(X_2,Y_2)}. \tag{2}$$

Let $Q = [0,1]^d$. For $r, m \in \mathbb{N}$ we let $P_m^{r,d,X} \in \mathscr{L}(C(Q,X))$ be composite with respect to the partition of $Q = [0,1]^d$ into m^d subcubes of sidelength m^{-1} tensor product Lagrange interpolation of degree r. Thus, $P_m^{r,d,X}$ interpolates on Γ_{rm}^d, where $\Gamma_k^d = \left\{ \frac{i}{k} : 0 \le i \le k \right\}^d$ for $k \in \mathbb{N}$. If $X = \mathbb{R}$, we write $P_m^{r,d}$. Note that in the sense of (1) we have $P_m^{r,d,X} = I_X \otimes P_m^{r,d}$. Furthermore, there are constants $c_1, c_2 > 0$ such that for all Banach spaces X and all m

$$\left\|P_m^{r,d,X}\right\|_{\mathscr{L}(C(Q,X))} \le c_1, \qquad \sup_{f\in B_{C^r(Q,X)}} \left\|f - P_m^{r,d,X} f\right\|_{C(Q,X)} \le c_2 m^{-r}. \tag{3}$$

This is well-known in the scalar case, for the easy extension to Banach spaces see [3].

3 Banach Space Valued Indefinite Integration

Let X be a Banach space and let the indefinite integration operator be given by

$$S_0^X : C([0,1],X) \to C([0,1],X), \quad (S_0^X f)(t) = \int_0^t f(\tau) d\tau \quad (t \in [0,1]).$$

First we recall the the Monte Carlo method from Section 4 of [13], here for integration domain $[0,1]$. Given $n \in \mathbb{N}$, we define $t_i = \frac{i}{n}$ $(0 \le i \le n)$. Let

$\xi_i : \Omega \to [t_i, t_{i+1}]$ be independent uniformly distributed random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$. For $f \in C(Q, X)$ and $\omega \in \Omega$ we define $g_\omega : \Gamma_n^1 \to \mathbb{R}$ by

$$g_\omega(t_i) = \frac{1}{n} \sum_{0 \le j < i} f(\xi_j(\omega)) \quad (0 \le i \le n).$$

Let $r \in \mathbb{N}_0$. If $r = 0$, we set

$$A_{n,\omega}^{0,0,X} f := P_n^{1,1,X} g_\omega.$$

and if $r \ge 1$,

$$A_{n,\omega}^{0,r,X} f = S_0^X (P_n^{r,1,X} f) + A_{n,\omega}^{0,0,X} (f - P_n^{r,1,X} f). \tag{4}$$

We write S_0 and $A_{n,\omega}^{0,r}$ if $X = \mathbb{R}$. Observe that in the sense of identification (1) we have

$$S_0^X = I_X \otimes S_0, \quad A_{n,\omega}^{0,r,X} = I_X \otimes A_{n,\omega}^{0,r}, \tag{5}$$

moreover, $A_{n,\omega}^{0,r,X} \in \mathscr{L}(C([0,1], X))$ and, since $g_\omega(0) = 0$,

$$\left(A_{n,\omega}^{0,r,X} f\right)(0) = 0 \quad (r \in \mathbb{N}_0). \tag{6}$$

We need the following result which complements Proposition 2 in [3].

Proposition 1. *Let $r \in \mathbb{N}_0$. Then there are constants $c_1, c_2 > 0$ such that for all Banach spaces X, $n \in \mathbb{N}$, $\omega \in \Omega$, and $f \in C([0,1], X)$ we have*

$$\|S_0^X f - A_{n,\omega}^{0,r,X} f\|_{C([0,1],X)} \le c_1 \|f\|_{C([0,1],X)} \tag{7}$$

$$(\mathbb{E}\, \|S_0^X f - A_{n,\omega}^{0,r,X} f\|_{C([0,1],X)}^2)^{1/2} \le c_2 \tau_2(X) n^{-1/2} \|f\|_{C([0,1],X)}. \tag{8}$$

Proof. Relation (7) directly follows from the definitions. By Proposition 2 in [3], there is a constant $c > 0$ such that for all Banach spaces X, $n \in \mathbb{N}$, and $f \in C([0,1], X)$ we have

$$\left(\mathbb{E}\, \|S_0^X f - A_{n,\omega}^{0,0,X} f\|_{C([0,1],X)}^2\right)^{1/2} \le c \tau_2(X) n^{-1/2} \|f\|_{C([0,1],X)}. \tag{9}$$

This is the case $r = 0$ of (8). Now assume $r \ge 1$. Then (3), (4), and (9) give

$$\begin{aligned} &\left(\mathbb{E}\, \|S_0^X f - A_{n,\omega}^{0,r,X} f\|_{C([0,1],X)}^2\right)^{1/2} \\ &= \left(\mathbb{E}\, \|S_0^X (f - P_n^{r,1,X} f) - A_{n,\omega}^{0,0,X} (f - P_n^{r,1,X} f)\|_{C([0,1],X)}^2\right)^{1/2} \\ &\le c \tau_2(X) n^{-1/2} \|f - P_n^{r,1,X} f\|_{C([0,1],X)} \le c \tau_2(X) n^{-1/2} \|f\|_{C([0,1],X)}. \end{aligned}$$

□

We shall further study the multilevel procedure developed in [3]. Let $(T_l)_{l=0}^\infty \subset \mathscr{L}(X)$. For convenience we introduce the following parameter set

$$\mathscr{M} = \left\{ \left(l_0, l_1, (n_{l_0})_{l=l_0}^{l_1}\right) : \, l_0, l_1 \in \mathbb{N}_0, \, l_0 \le l_1, \, (n_{l_0})_{l=l_0}^{l_1} \subset \mathbb{N} \right\}. \tag{10}$$

For $\mu \in \mathscr{M}$ we define an approximation $A^{0,r}_{\mu,\omega}$ to S^X_0 as follows:

$$A^{0,r}_{\mu,\omega} = T_{l_0} \otimes A^{0,r}_{n_{l_0},\omega} + \sum_{l=l_0+1}^{l_1} (T_l - T_{l-1}) \otimes A^{0,0}_{n_l,\omega}, \tag{11}$$

where the tensor product is understood in the sense of (1). We assume that the random variables $A^{0,r}_{n_{l_0},\omega}$ and $\left(A^{0,0}_{n_l,\omega}\right)_{l=l_0+1}^{l_1}$ are independent. We have

$$A^{0,r}_{\mu,\omega} \in \mathscr{L}(C([0,1],X)).$$

Denote

$$X_l = \mathrm{cl}_X(T_l(X)) \quad (l \in \mathbb{N}_0) \tag{12}$$

$$X_{l-1,l} = \mathrm{cl}_X((T_l - T_{l-1})(X)) \quad (l \in \mathbb{N}), \tag{13}$$

where cl_X denotes the closure in X. In particular, X_l and $X_{l-1,l}$ are endowed with the norm induced by X. The following result complements Proposition 3 in [3].

Proposition 2. *There is a constant $c > 0$ such that for all Banach spaces X, and operators $(T_l)_{l=0}^{\infty}$ as above, for all $\mu \in \mathscr{M}$*

$$\begin{aligned}&\sup_{f \in B_{C([0,1],X)}} \left(\mathbb{E}\, \|S^X_0 f - A^{0,r}_{\mu,\omega} f\|^2_{C([0,1],X)}\right)^{1/2} \\ &\leq \|I_X - T_{l_1}\|_{\mathscr{L}(X)} + c\tau_2(X_{l_0})\|T_{l_0}\|_{\mathscr{L}(X)} n_{l_0}^{-1/2} \\ &\quad + c \sum_{l=l_0+1}^{l_1} \tau_2(X_{l-1,l})\|(T_l - T_{l-1})\|_{\mathscr{L}(X)} n_l^{-1/2}. \end{aligned} \tag{14}$$

Proof. Denote

$$R_l = T_l \otimes I_{C([0,1])} \in \mathscr{L}(C([0,1],X)). \tag{15}$$

From (5) and (11) we get

$$A^{0,r}_{\mu,\omega} = A^{0,r,X}_{n_{l_0},\omega} R_{l_0} + \sum_{l=l_0+1}^{l_1} A^{0,0,X}_{n_l,\omega}(R_l - R_{l-1}). \tag{16}$$

To prove (14), let $f \in B_{C([0,1],X)}$. Then by (16),

$$\begin{aligned}&\|S^X_0 f - A^{0,r}_{\mu,\omega} f\|_{C([0,1],X)} \\ &\leq \|S^X_0 f - S^X_0 R_{l_1} f\|_{C([0,1],X)} + \|S^X_0 R_{l_0} f - A^{0,r,X}_{n_{l_0},\omega} R_{l_0} f\|_{C([0,1],X_{l_0})} \\ &\quad + \left\| \sum_{l=l_0+1}^{l_1} \left(S^X_0 (R_l - R_{l-1}) f - A^{0,0,X}_{n_l,\omega}(R_l - R_{l-1}) f\right) \right\|_{C([0,1],X_{l-1,l})}. \end{aligned} \tag{17}$$

We have, using (2),

$$\begin{aligned}&\|S_0^X f - S_0^X R_{l_1} f\|_{C([0,1],X)} \le \|S_0^X\|_{\mathscr{L}(C([0,1],X))} \|f - R_{l_1} f\|_{C([0,1],X)} \\ \le\ & \|I_X - T_{l_1}\|_{\mathscr{L}(X)} \|f\|_{C([0,1],X)} \le \|I_X - T_{l_1}\|_{\mathscr{L}(X)}. \qquad (18)\end{aligned}$$

Furthermore, by Proposition 1

$$\begin{aligned}&\mathbb{E}\left(\|S_0^X R_{l_0} f - A^{0,r,X}_{n_{l_0},\omega} R_{l_0} f\|^2_{C([0,1],X_{l_0})}\right)^{1/2} \\ \le\ & c\tau_2(X_{l_0}) \|T_{l_0}\|_{\mathscr{L}(X)} n_{l_0}^{-1/2}. \qquad (19)\end{aligned}$$

For $l_0 < l \le l_1$ we obtain

$$\begin{aligned}&\mathbb{E}\left(\|S_0^X (R_l - R_{l-1}) f - A^{0,0,X}_{n_l,\omega} (R_l - R_{l-1}) f\|^2_{C([0,1],X_{l-1,l})}\right)^{1/2} \\ \le\ & c\tau_2(X_{l-1,l}) \|T_l - T_{l-1}\|_{\mathscr{L}(X)} n_l^{-1/2}. \qquad (20)\end{aligned}$$

Combining (17–20) yields the result. □

4 Parametric Indefinite Integration

Let $d \in \mathbb{N}$, $Q = [0,1]^d$. The indefinite parametric integration operator $S_1 : C(Q \times [0,1]) \to C(Q \times [0,1])$ is given by

$$(S_1 f)(s,t) = \int_0^t f(s,\tau)d\tau \quad (s \in Q, t \in [0,1]).$$

This problem is related to the Banach space case from the previous section as follows. With $X = C(Q)$ we have the identifications

$$C(Q \times [0,1]) = C([0,1],X), \quad S_1 = S_0^{C(Q)}.$$

Let $r_1 = \max(r,1)$ and define for $l \in \mathbb{N}_0$

$$T_l = P^{r_1,d}_{2^l} \in \mathscr{L}(C(Q)). \qquad (21)$$

By (3),

$$\|T_l\|_{\mathscr{L}(C(Q))} \le c_1, \quad \|J - T_l J\|_{\mathscr{L}(C^r(Q),C(Q))} \le c_2 2^{-rl}, \qquad (22)$$

where $J : C^r(Q) \to C(Q)$ is the embedding. For $\mu = \left(l_0, l_1, (n_{l_0})_{l=l_0}^{l_1}\right) \in \mathscr{M}$ the algorithm $A^{0,r}_{\mu,\omega}$ defined in (11) takes the following form. For $f \in C(Q \times [0,1])$

$$\begin{aligned}A^{1,r}_{\mu,\omega} f &= P^{r_1,d}_{2^{l_0}}\left(\left(A^{0,r}_{n_{l_0},\omega}(f_s)\right)_{s\in\Gamma^d_{r_1 2^{l_0}}}\right) \\ &+ \sum_{l=l_0+1}^{l_1} \left(P^{r_1,d}_{2^l} - P^{r_1,d}_{2^{l-1}}\right)\left(\left(A^{0,0}_{n_l,\omega}(f_s)\right)_{s\in\Gamma^d_{r_1 2^l}}\right), \qquad (23)\end{aligned}$$

where for $s \in Q$ we used the notation $f_s = f(s,\,\cdot\,)$. Then

$$\operatorname{card}\left(A^{1,r}_{\mu,\omega}\right) \le c \sum_{l=l_0}^{l_1} n_l 2^{dl} \quad (\omega \in \Omega), \tag{24}$$

where $\operatorname{card}\left(A^{1,r}_{\mu,\omega}\right)$ denotes the cardinality, that is, the number of function values used in algorithm $A^{1,r}_{\mu,\omega}$ (see also the general remarks before Theorem 1 below). Moreover, we have $A^{1,r}_{\mu,\omega} \in \mathscr{L}(C(Q \times [0,1]))$ and it follows from (6) that

$$\left(A^{1,r}_{\mu,\omega} f\right)(s,0) = 0 \quad (s \in Q). \tag{25}$$

We also consider the following subset $\mathscr{M}_0 \subset \mathscr{M}$ corresponding to one-level algorithms

$$\mathscr{M}_0 = \{\mu \in \mathscr{M} : \mu = (l_0, l_0, n_{l_0})\} \tag{26}$$

thus, for $\mu_0 \in \mathscr{M}_0$,

$$A^{1,r}_{\mu_0,\omega} f \quad = \quad P^{r_1,d}_{2^{l_0}}\left(\left(A^{0,r}_{n_{l_0},\omega}(f_s)\right)_{s \in \Gamma^d_{r_1 2^{l_0}}}\right). \tag{27}$$

Parts of the following result (relations (30) and (33)) were shown in [3], Proposition 4. We prove that algorithm $A^{1,r}_{\mu,\omega}$ simultaneously satisfies the estimates (32) and (33). The former is crucial for the stability analysis of the iteration in Section 5. We note that, due to the multilevel structure of $A^{1,r}_{\mu,\omega}$ relation (32) is not trivial (a trivial estimate would be $c\log(n+1)$). Some of the choices of multilevel parameters from [3] are not suitable to obtain both estimate simultaneously. So here we provide modified choices and verify the needed estimates for them, still using the analysis of [3].

Proposition 3. *Let $r \in \mathbb{N}_0$, $d \in \mathbb{N}$. There are constants $c_{1-6} > 0$ such that the following hold. For each $n \in \mathbb{N}$ there is a $\mu_0(n) \in \mathscr{M}_0$ such that for all $\omega \in \Omega$*

$$\operatorname{card}\left(A^{1,r}_{\mu_0(n),\omega}\right) \quad \le \quad c_1 n \tag{28}$$

$$\sup_{f \in B_{C(Q\times[0,1])}} \|S_1 f - A^{1,r}_{\mu_0(n),\omega} f\|_{C(Q\times[0,1])} \quad \le \quad c_2 \tag{29}$$

$$\sup_{f \in B_{C^r(Q\times[0,1])}} \|S_1 f - A^{1,r}_{\mu_0(n),\omega} f\|_{C(Q\times[0,1])} \quad \le \quad c_3 n^{-\frac{r}{d+1}}. \tag{30}$$

Moreover, for each $n \in \mathbb{N}$ there is a $\mu(n) \in \mathscr{M}$ such that

$$\max_{\omega \in \Omega} \operatorname{card}\left(A^{1,r}_{\mu(n),\omega}\right) \quad \le \quad c_4 n \tag{31}$$

$$\sup_{f \in B_{C(Q\times[0,1])}} \left(\mathbb{E}\,\|S_1 f - A^{1,r}_{\mu(n),\omega} f\|^2_{C(Q\times[0,1])}\right)^{1/2} \quad \le \quad c_5 \tag{32}$$

$$\sup_{f \in B_{C^r(Q\times[0,1])}} \left(\mathbb{E}\,\|S_1 f - A^{1,r}_{\mu(n),\omega} f\|^2_{C(Q\times[0,1])}\right)^{1/2} \quad \le \quad c_6 n^{-\gamma_1} (\log(n+1))^{\gamma_2} \tag{33}$$

with

$$\gamma_1 = \begin{cases} \frac{r+1/2}{d+1} & \text{if} \quad \frac{r}{d} > \frac{1}{2} \\ \frac{r}{d} & \text{if} \quad \frac{r}{d} \le \frac{1}{2} \end{cases} \qquad \gamma_2 = \begin{cases} \frac{1}{2} & \text{if} \quad \frac{r}{d} > \frac{1}{2} \\ 2 & \text{if} \quad \frac{r}{d} = \frac{1}{2} \\ \frac{r}{d} & \text{if} \quad \frac{r}{d} < \frac{1}{2}\,. \end{cases} \tag{34}$$

Proof. Let $n \in \mathbb{N}$, put

$$l^* = \left\lceil \frac{\log_2(n+1)}{d} \right\rceil, \quad l_0 = \left\lfloor \frac{d}{d+1} l^* \right\rfloor, \quad n_{l_0} = 2^{d(l^*-l_0)} \tag{35}$$

and $\mu_0(n) = (l_0, l_0, n_{l_0})$. For this choice relations (28) and (30) were shown in [3]. Relation (29) readily follows from (3), (7) of Proposition 1, and (27).

To prove (31–34), let $\mu(n) = \big(l_0, l_1, (n_{l_0})_{l=l_0}^{l_1}\big) \in \mathscr{M}$, with l_0 and n_{l_0} given by (35), and l_1, $(n_{l_0})_{l=l_0+1}^{l_1}$ to be fixed later on. For brevity we denote for $\varrho \in \mathbb{N}_0$,

$$E_\varrho(\mu(n)) := \sup_{f \in B_{C^\varrho(Q\times[0,1])}} \Big(\mathbb{E}\, \|S_1 f - A^{1,r}_{\mu(n),\omega} f\|^2_{C(Q\times[0,1])} \Big)^{1/2}.$$

We show that for $\varrho \in \{0, r\}$

$$E_\varrho(\mu(n)) \;\le\; c2^{-\varrho l_1} + c(l_0+1)^{1/2} n_{l_0}^{-\varrho-1/2} + c \sum_{l=l_0+1}^{l_1} (l+1)^{1/2} 2^{-\varrho l} n_l^{-1/2}. \tag{36}$$

By (63) in [3], this holds for $\varrho = r$. It remains to prove the corresponding estimate for $r \ge 1$, $\varrho = 0$. By (12) and (21)

$$X_l = P^{r_1,d}_{2^l}(C(Q)), \tag{37}$$

therefore $X_{l-1} \subseteq X_l$ and, by (13), also $X_{l-1,l} \subseteq X_l$ for $l \ge 1$. As shown in [3],

$$\tau_2(X_{l-1,l}) \le \tau_2(X_l) \le c(l+1)^{1/2}. \tag{38}$$

We conclude from (14) of Proposition 2, (22), and (38) that

$$E_0(\mu(n)) \;\le\; c + c(l_0+1)^{1/2} n_{l_0}^{-1/2} + c \sum_{l=l_0+1}^{l_1} (l+1)^{1/2} n_l^{-1/2}, \tag{39}$$

which shows (36) for $\varrho = 0$.

From (35) we conclude

$$d(l^* - l_0) \ge \frac{dl^*}{d+1} \ge l_0,$$

thus,

$$n_{l_0}^{-\varrho-1/2} = 2^{-(\varrho+1/2)d(l^*-l_0)} \le 2^{-\varrho l_0 - d(l^*-l_0)/2} = 2^{-\varrho l_0} n_{l_0}^{-1/2}.$$

This means that we can include the middle term in (36) into the sum, which gives

$$E_\varrho(\mu(n)) \;\leq\; c2^{-\varrho l_1} + c\sum_{l=l_0}^{l_1}(l+1)^{1/2}2^{-\varrho l}n_l^{-1/2} \quad (\varrho \in \{0, r\}). \tag{40}$$

If $r > d/2$, we set

$$\gamma = \frac{(r+1/2)d}{r(d+1)}, \quad l_1 = \lceil \gamma l^* \rceil \,.$$

Then

$$\frac{d}{d+1} < \gamma < 1.$$

Indeed, the left hand inequality is obvious, while the right-hand inequality is a consequence of the assumption $r > d/2$. With (35) it follows that

$$l_0 \leq l_1 \leq l^*.$$

We choose a $\delta > 0$ in such a way that

$$r - \delta/2 \;>\; d/2, \tag{41}$$

$$\delta\left(\gamma - \frac{d}{d+1}\right) \;<\; d(1-\gamma) \tag{42}$$

and put

$$n_l \;=\; \left\lceil 2^{d(l^*-l)-\delta(l-l_0)} \right\rceil \quad (l = l_0+1, \ldots, l_1).$$

From (40–42) and (35) we obtain

$$\begin{aligned} E_r(\mu(n)) \;&\leq\; c2^{-rl_1} + c(l^*+1)^{1/2}\sum_{l=l_0}^{l_1} 2^{-rl_0-(r-\delta/2)(l-l_0)-d(l^*-l)/2} \\ &\leq\; c2^{-\frac{(r+1/2)d}{d+1}l^*} + c(l^*+1)^{1/2}2^{-rl_0-d(l^*-l_0)/2} \\ &\leq\; c(l^*+1)^{1/2}2^{-\frac{(r+1/2)d}{d+1}l^*} \leq cn^{-\frac{r+1/2}{d+1}}(\log(n+1))^{1/2}. \end{aligned}$$

Furthermore, using (40) and (42),

$$\begin{aligned} E_0(\mu(n)) \;&\leq\; c + c(l^*+1)^{1/2}\sum_{l=l_0}^{l_1} 2^{\delta(l-l_0)/2-d(l^*-l)/2} \\ &\leq\; c + c(l^*+1)^{1/2}2^{\delta(l_1-l_0)/2-d(l^*-l_1)/2} \\ &\leq\; c + c(l^*+1)^{1/2}2^{\delta\left(\gamma-\frac{d}{d+1}\right)\frac{l^*}{2}-d(1-\gamma)\frac{l^*}{2}} \leq c. \end{aligned}$$

By (24) the number of function values fulfills

$$\operatorname{card}\left(A^{1,r}_{\mu(n),\omega}\right) \;\leq\; c\sum_{l=l_0}^{l_1} n_l 2^{dl} \leq c2^{dl_1} + c\sum_{l=l_0}^{l_1} 2^{dl^*-\delta(l-l_0)} \leq cn. \tag{43}$$

This proves (31–34) for $r > d/2$.

If $r = d/2$, we set $l_1 = l^*$, put

$$n_l = \max\left(2^{d(l^*-l)}, \lceil (l^*+1)2^{d(l^*-l)/2} \rceil\right) \quad (l = l_0+1, \dots, l_1)$$

and get from (35) and (40),

$$\begin{aligned} E_r(\mu(n)) &\leq c2^{-rl^*} + c(l^*+1)^{1/2} \sum_{l=l_0}^{l^*} 2^{-rl-d(l^*-l)/2} \\ &\leq c(l^*+1)^{3/2} 2^{-dl^*/2} \leq cn^{-1/2}(\log(n+1))^{3/2}, \end{aligned} \tag{44}$$

$$E_0(\mu(n)) \leq c + c(l^*+1)^{1/2} \sum_{l=l_0}^{l_1} n_l^{-1/2} \leq c + c\sum_{l=l_0}^{l^*} 2^{-d(l^*-l)/4} \leq c. \tag{45}$$

The cardinality satisfies

$$\begin{aligned} \operatorname{card}\left(A^{1,r}_{\mu(n),\omega}\right) &\leq c\sum_{l=l_0}^{l^*} n_l 2^{dl} \leq c2^{dl^*} + c\sum_{l=l_0}^{l^*}\left(2^{dl^*} + (l^*+1)2^{d(l^*+l)/2}\right) \\ &\leq c(l^*+1)2^{dl^*} \leq cn\log(n+1). \end{aligned} \tag{46}$$

Transforming $n\log(n+1)$ into n in relations (44–46) proves (31–34) for this case.

Finally, if $r < d/2$, we set

$$l_1 = l^* - \lceil d^{-1}\log_2(l^*+1) \rceil, \tag{47}$$

choose a $\delta > 0$ in such a way that

$$(d-\delta)/2 > r \tag{48}$$

and put

$$n_l = \lceil 2^{d(l^*-l)-\delta(l_1-l)} \rceil \quad (l = l_0+1, \dots, l_1). \tag{49}$$

This is the same choice as in the respective case of the proof of Proposition 4 in [3]. Clearly, there is a constant $c > 0$ such that $l_0 \leq l_1$ for $n \geq c$. For $n < c$ the statements (32) and (33) are trivial. It was shown in [3] that with the choice above $\operatorname{card}\left(A^{1,r}_{\mu(n),\omega}\right) \leq cn$ and that relation (33) holds. Arguing similarly, we derive from (35), (36), (47), and (49) for the case $\varrho = 0$

$$\begin{aligned} E_0(\mu(n)) &\leq c + c(l^*+1)^{1/2}2^{-d(l^*-l_0)/2} \\ &\quad + c(l^*+1)^{1/2}\sum_{l=l_0+1}^{l_1} 2^{-d(l^*-l_1)/2-(d-\delta)(l_1-l)/2} \\ &\leq c + c(l^*+1)^{1/2}2^{-d(l^*-l_1)/2} \\ &\leq c + c(l^*+1)^{1/2}2^{-(\log_2(l^*+1))/2} \leq c, \end{aligned}$$

which is (32).

□

5 Fixed Point Iteration for Parametric ODEs

Here we apply the above results to the following problem. Let $d, q \in \mathbb{N}$, $r \in \mathbb{N}_0$, $Q = [0,1]^d$, and let $C^r_{\mathrm{Lip}}(Q \times [0,1] \times \mathbb{R}^q, \mathbb{R}^q)$ be the space of functions $f \in C^r(Q \times [0,1] \times \mathbb{R}^q, \mathbb{R}^q)$ satisfying for $s \in Q$, $t \in [0,1]$, $z_1, z_2 \in \mathbb{R}^q$

$$|f|_{\mathrm{Lip}} := \sup_{s \in Q, t \in [0,1], z_1 \neq z_2 \in \mathbb{R}^q} \frac{\|f(s,t,z_1) - f(s,t,z_2)\|_{\mathbb{R}^q}}{\|z_1 - z_2\|_{\mathbb{R}^q}} < \infty. \tag{50}$$

The space $C^r_{\mathrm{Lip}}(Q \times [0,1] \times \mathbb{R}^q, \mathbb{R}^q)$ is endowed with the norm

$$\|f\|_{C^r_{\mathrm{Lip}}(Q \times [0,1] \times \mathbb{R}^q, \mathbb{R}^q)} = \max\left(\|f\|_{C^r(Q \times [0,1] \times \mathbb{R}^q, \mathbb{R}^q)}, |f|_{\mathrm{Lip}}\right). \tag{51}$$

If $r = 0$, we also write $C_{\mathrm{Lip}}(Q \times [0,1] \times \mathbb{R}^q, \mathbb{R}^q)$. We consider the numerical solution of initial value problems for systems of ODEs depending on a parameter $s \in Q$

$$\frac{\partial u(s,t)}{\partial t} = f(s,t,u(s,t)) \quad (s \in Q, t \in [0,1]) \tag{52}$$

$$u(s,0) = u_0(s) \quad (s \in Q) \tag{53}$$

with $f \in C_{\mathrm{Lip}}(Q \times [0,1] \times \mathbb{R}^q, \mathbb{R}^q)$ and $u_0 \in C(Q, \mathbb{R}^q)$. A function $u : Q \times [0,1] \to \mathbb{R}^q$ is called a solution if for each $s \in Q$, $u(s,t)$ is continuously differentiable as a function of t and (52–53) are satisfied. Due to the assumptions on f and u_0 the solution exists, is unique, and belongs to $C(Q \times [0,1], \mathbb{R}^q)$. Let the solution operator

$$S_2 : C_{\mathrm{Lip}}(Q \times [0,1] \times \mathbb{R}^q, \mathbb{R}^q) \times C(Q, \mathbb{R}^q) \to C(Q \times [0,1], \mathbb{R}^q)$$

be given by $S_2(f, u_0) = u$, where $u = u(s,t)$ is the solution of (52–53). Furthermore, fix $\kappa > 0$ and let

$$F^r_2(\kappa) = \kappa B_{C^r_{\mathrm{Lip}}(Q \times [0,1] \times \mathbb{R}^q, \mathbb{R}^q)} \times \kappa B_{C^r(Q, \mathbb{R}^q)}. \tag{54}$$

Classical results on the regularity with respect to t and the parameter s (see, e.g., [23]) give

$$\sup_{(f,u_0) \in F^r_2(\kappa)} \|S_2(f, u_0)\|_{C^r(Q \times [0,1], \mathbb{R}^q)} \leq c. \tag{55}$$

Now let $f \in C_{\mathrm{Lip}}(Q \times [0,1] \times \mathbb{R}^q, \mathbb{R}^q)$ and $u_0 \in C(Q, \mathbb{R}^q)$. We rewrite (52–53) in the equivalent form

$$u(s,t) = u_0(s) + \int_0^t f(s,\tau,u(s,\tau))d\tau \quad (s \in Q, t \in [0,1]). \tag{56}$$

Let $m \in \mathbb{N}$ and $t_i = im^{-1}$ $(i = 0, \dots, m)$. We solve (56) and thus (52–53) in m steps on the intervals $[t_i, t_{i+1}]$ $(i = 0, \dots, m-1)$. Let

$$S_{1,i} : C(Q \times [t_i, t_{i+1}], \mathbb{R}^q) \to C(Q \times [t_i, t_{i+1}], \mathbb{R}^q)$$

be the q-dimensional version of the solution operator of parametric indefinite integration on $[t_i, t_{i+1}]$, i.e., for $g \in C(Q \times [t_i, t_{i+1}], \mathbb{R}^q)$

$$\big(S_{1,i} g\big)(s,t) = \left(\int_{t_i}^{t} g_l(s,\tau) d\tau \right)_{l=1}^{q} \quad (t \in [t_i, t_{i+1}]), \tag{57}$$

where g_l are the components of g. Let $A^{1,r}_{\mu,i,\omega}$ be algorithm $A^{1,r}_{\mu,\omega}$ from (23), scaled to $[t_i, t_{i+1}]$ and applied to each component of g, that is

$$\big(A^{1,r}_{\mu,i,\omega} g\big)(s,t) = \Big(m^{-1} \big(A^{1,r}_{\mu,\omega} g_l^*\big)(s, m(t - t_i)) \Big)_{l=1}^{q} \quad (t \in [t_i, t_{i+1}]), \tag{58}$$

with

$$g_l^*(s,\tau) = g_l(s, t_i + m^{-1}\tau) \quad (\tau \in [0,1]). \tag{59}$$

Let

$$\mathscr{N} \;=\; \big\{ \big(m, M, k, (\mu_j)_{j=0}^{k-1}\big) : \, m, M, k \in \mathbb{N}, \, (\mu_j)_{j=0}^{k-1} \subset \mathscr{M} \big\} \tag{60}$$

$$\mathscr{N}_0 \;=\; \big\{ \big(m, M, k, (\mu_j)_{j=0}^{k-1}\big) \in \mathscr{N} : \, (\mu_j)_{j=0}^{k-1} \subset \mathscr{M}_0 \big\} \subset \mathscr{N}, \tag{61}$$

where $\mathscr{M}$ and $\mathscr{M}_0$ were defined in (10) and (26), respectively, and let $r_1 = \max(r, 1)$. For $\nu = \big(m, M, k, (\mu_j)_{j=0}^{k-1}\big) \in \mathscr{N}$ define $u_{0,0} = P_M^{r_1,d} u_0$ and for $i = 0, \dots, m-1$, $j = 0, \dots, k-1$, $s \in Q$ the iteration

$$u_{i,j+1}(s,t) = u_{i,0}(s) + (A^{1,r}_{\mu_j,i,\omega} g_{ij})(s,t) \quad (t \in [t_i, t_{i+1}]), \tag{62}$$

where

$$g_{ij}(s,t) = f(s, t, u_{ij}(s,t)) \quad (t \in [t_i, t_{i+1}]), \tag{63}$$

and

$$u_{i+1,0}(s) = u_{ik}(s, t_{i+1}) \qquad (t \in [t_{i+1}, t_{i+2}], \, i \le m-2). \tag{64}$$

We assume that the involved random variables $(A^{1,r}_{\mu_j,i,\omega})_{i,j=0}^{m-1,k-1}$ are independent. Furthermore, for $s \in Q$, $t \in [0,1]$ put

$$u_\nu(s,t) \;=\; \begin{cases} u_{ik}(s,t) & \text{if} \quad t \in [t_i, t_{i+1}), \ i \le m-2 \\ u_{m-1,k}(s,t) & \text{if} \quad t \in [t_{m-1}, t_m] \end{cases} \tag{65}$$

$$A^{2,r}_{\nu,\omega}(f, u_0) \;=\; u_\nu. \tag{66}$$

Clearly, $u_{ij} \in C(Q \times [t_i, t_{i+1}], \mathbb{R}^q)$. Moreover, it follows from (58) and (25) that

$$(A^{1,r}_{\mu_j,i,\omega} g_{ij})(s, t_i) = 0,$$

and therefore (62) yields

$$u_{ik}(s, t_i) = u_{i,0}(s) = u_{i-1,k}(s, t_i) \quad (1 \le i \le m-1, \, s \in Q),$$

hence $v \in C(Q \times [0,1], \mathbb{R}^q)$. Next we give error and stability estimates for $A^{2,r}_{\nu,\omega}$.

Proposition 4. *Let $r \in \mathbb{N}_0$, $d, q \in \mathbb{N}$, $\kappa > 0$. Then there are constants $c_{1-6} > 0$ such that the following hold. For each $n \in \mathbb{N}$ there is a $\nu_0(n) \in \mathscr{N}_0$ such that for all $\omega \in \Omega$*

$$\operatorname{card}\left(A^{2,r}_{\nu_0(n),\omega}\right) \leq c_1 n \tag{67}$$

$$\sup_{(f,u_0)\in F_2^r(\kappa)} \|S_2(f,u_0) - A^{2,r}_{\nu_0(n),\omega}(f,u_0)\|_{C(Q\times[0,1],\mathbb{R}^q)} \leq c_2 n^{-\frac{r}{d+1}} \tag{68}$$

and for $(f,u_0), (\tilde{f},\tilde{u}_0) \in F_2^0(\kappa)$

$$\begin{aligned}&\|A^{2,r}_{\nu_0(n),\omega}(f,u_0) - A^{2,r}_{\nu_0(n),\omega}(\tilde{f},\tilde{u}_0)\|_{C(Q\times[0,1],\mathbb{R}^q)}\\ &\quad\leq c_3\big(\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)} + \|u_0-\tilde{u}_0\|_{C(Q,\mathbb{R}^q)}\big).\end{aligned} \tag{69}$$

Moreover, for each $n \in \mathbb{N}$ there is a $\nu(n) \in \mathscr{N}$ such that

$$\max_{\omega\in\Omega} \operatorname{card}\left(A^{2,r}_{\nu(n),\omega}\right) \leq c_4 n, \tag{70}$$

$$\begin{aligned}&\sup_{(f,u_0)\in F_2^r(\kappa)} \left(\mathbb{E}\,\|S_2(f,u_0) - A^{2,r}_{\nu(n),\omega}(f,u_0)\|^2_{C(Q\times[0,1],\mathbb{R}^q)}\right)^{1/2}\\ &\quad\leq c_5 n^{-\gamma_1}(\log(n+1))^{\gamma_2},\end{aligned} \tag{71}$$

with γ_1 and γ_2 given by (34), and for $(f,u_0), (\tilde{f},\tilde{u}_0) \in F_2^0(\kappa)$

$$\begin{aligned}&\left(\mathbb{E}\,\|A^{2,r}_{\nu(n),\omega}(f,u_0) - A^{2,r}_{\nu(n),\omega}(\tilde{f},\tilde{u}_0)\|^2_{C(Q\times[0,1],\mathbb{R}^q)}\right)^{1/2}\\ &\quad\leq c_6\big(\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)} + \|u_0-\tilde{u}_0\|_{C(Q,\mathbb{R}^q)}\big).\end{aligned} \tag{72}$$

Proof. We prove (70–72), the proof of (67–69) is analogous, just simpler. For the sake of brevity we set

$$\varepsilon(n) = n^{-\gamma_1}(\log(n+1))^{\gamma_2}. \tag{73}$$

By Proposition 3 and (57–59) there are constants $c, c(1), c(2) > 0$ and a sequence $(\mu(n))_{n=1}^{\infty} \subset \mathscr{M}$ such that for $m, n \in \mathbb{N}$

$$\max_{\omega\in\Omega} \operatorname{card}\left(A^{1,r}_{\mu(n),i,\omega}\right) \leq cn, \tag{74}$$

for $f \in C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)$

$$\left(\mathbb{E}\,\|S_{1,i}f - A^{1,r}_{\mu(n),i,\omega}f\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2} \leq c(1)m^{-1}\|f\|_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)} \tag{75}$$

and for $f \in C^r(Q\times[t_i,t_{i+1}],\mathbb{R}^q)$

$$\begin{aligned}&\left(\mathbb{E}\,\|S_{1,i}f - A^{1,r}_{\mu(n),i,\omega}f\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\ &\quad\leq c(2)m^{-1}\varepsilon(n)\|f\|_{C^r(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}.\end{aligned} \tag{76}$$

In the rest of the proof we reserve the notation $c(1)$ and $c(2)$ for the constants in (75) and (76). We need the following stability property, which is a consequence of (75) and the linearity of $A_{\mu(n),i,\omega}$: For $f_1, f_2 \in C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)$

$$\begin{aligned}
&\left(\mathbb{E}\,\|A^{1,r}_{\mu(n),i,\omega}f_1 - A^{1,r}_{\mu(n),i,\omega}f_2\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\le \|S_{1,i}(f_1-f_2)\|_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\\
&\quad + \left(\mathbb{E}\,\|S_{1,i}(f_1-f_2) - A^{1,r}_{\mu(n),i,\omega}(f_1-f_2)\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\le (c(1)+1)m^{-1}\|f_1-f_2\|_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}. \qquad (77)
\end{aligned}$$

We choose $m\in\mathbb{N}$ in such a way that

$$\theta := (c(1)+1)m^{-1}\kappa \le 1/2. \qquad (78)$$

Now we fix $n\in\mathbb{N}$ and define

$$M = \lceil n^{1/d}\rceil, \quad k = \left\lfloor \frac{\gamma_1\log_2 n + \log_2 m}{-\log_2\theta}\right\rfloor + 1, \qquad (79)$$

$$n_j = \left\lceil n\theta^{\frac{k-1-j}{\gamma_1+1}}\right\rceil \quad (j=0,\dots,k-1), \qquad (80)$$

and set

$$\nu(n) = \left(m, M, k, \mu(n_j)_{j=0}^{k-1}\right).$$

Then the cardinality of algorithm $A^{2,r}_{\nu(n),\omega}$ satisfies

$$\operatorname{card}\left(A^{2,r}_{\nu(n),\omega}\right) \le cM^d + cm\sum_{j=0}^{k-1} n_j \le cn + cm\sum_{j=0}^{k-1}\left\lceil n\theta^{\frac{k-1-j}{\gamma_1+1}}\right\rceil \le cn$$

(note that by the choice (78), m is just a constant). This shows (70).

Next we prove the error estimate (71). Let $(f,u_0)\in F_2^r(\kappa)$. By (3) and (34)

$$\|u(\,\cdot\,,0)-u_{0,0}\|_{C(Q,\mathbb{R}^q)} = \left\|u_0 - P_M^{r_1,d}u_0\right\|_{C(Q,\mathbb{R}^q)} \le cn^{-r/d} \le cn^{-\gamma_1}. \qquad (81)$$

Setting

$$g(s,t) = f(s,t,u(s,t)), \qquad (82)$$

we get from (55)

$$\|g\|_{C^r(Q\times[0,1],\mathbb{R}^q)} \le c. \qquad (83)$$

Moreover, (63) implies

$$\|g-g_{ij}\|_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)} \le \kappa\|u-u_{ij}\|_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}. \qquad (84)$$

We have

$$u(s,t) = u(s,t_i) + (S_{1,i}g)(s,t) \quad (s\in Q, t\in[t_i,t_{i+1}]).$$

We estimate, using (83), (84), (76), and (77)

$$
\begin{aligned}
&\left(\mathbb{E}\,\|u-u_{i,j+1}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\le \left(\mathbb{E}\,\left\|u(\,\cdot\,,t_i)+S_{1,i}g-u_{i,0}-A^{1,r}_{\mu(n_j),i,\omega}g_{ij}\right\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\le \left(\mathbb{E}\,\|u(\,\cdot\,,t_i)-u_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2}+\left(\mathbb{E}\,\left\|S_{1,i}g-A^{1,r}_{\mu(n_j),i,\omega}g\right\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\quad+\left(\mathbb{E}\,\mathbb{E}\,\left(\left\|A^{1,r}_{\mu(n_j),i,\omega}g-A^{1,r}_{\mu(n_j),i,\omega}g_{ij}\right\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\Big|u_{ij}\right)\right)^{1/2}\\
&\le \left(\mathbb{E}\,\|u(\,\cdot\,,t_i)-u_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2}+\left(\mathbb{E}\,\left\|S_{1,i}g-A^{1,r}_{\mu(n_j),i,\omega}g\right\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\quad+(c(1)+1)m^{-1}\left(\mathbb{E}\,\|g-g_{ij}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\le \left(\mathbb{E}\,\|u(\,\cdot\,,t_i)-u_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2}+c(1)m^{-1}\varepsilon(n_j)\\
&\quad+\theta\left(\mathbb{E}\,\|u-u_{ij}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}. \qquad (85)
\end{aligned}
$$

We get from (85) by recursion over j

$$
\begin{aligned}
&\left(\mathbb{E}\,\|u-u_{ik}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\le \left(\mathbb{E}\,\|u(\,\cdot\,,t_i)-u_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2}\sum_{j=0}^{k-1}\theta^j+c(1)m^{-1}\sum_{j=0}^{k-1}\theta^j\varepsilon(n_{k-j-1})\\
&\quad+\theta^k\left(\mathbb{E}\,\|u-u_{i,0}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\le \theta^k\,\|u-u(\,\cdot\,,t_i)\|_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}+\left(\mathbb{E}\,\|u(\,\cdot\,,t_i)-u_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2}\sum_{j=0}^{k}\theta^j\\
&\quad+c(1)m^{-1}\sum_{j=0}^{k-1}\theta^j\varepsilon(n_{k-j-1}). \qquad (86)
\end{aligned}
$$

By (55) and (79),

$$\theta^k\,\|u-u(\,\cdot\,,t_i)\|_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\le c\theta^k\le cm^{-1}n^{-\gamma_1}. \qquad (87)$$

Moreover, (78) implies

$$\sum_{j=0}^{k}\theta^j=\left(1+\theta\frac{1-\theta^k}{1-\theta}\right)\le 1+2\theta. \qquad (88)$$

Finally, using (80) and (73), we obtain

$$\begin{aligned}\sum_{j=0}^{k-1}\theta^j\varepsilon(n_{k-j-1}) &= \sum_{j=0}^{k-1}\theta^j n_{k-j-1}^{-\gamma_1}(\log(n_{k-j-1}+1))^{\gamma_2}\\ &\le \sum_{j=0}^{k-1}\theta^j n^{-\gamma_1}\theta^{-\frac{\gamma_1 j}{\gamma_1+1}}(\log(n+1))^{\gamma_2}\\ &= n^{-\gamma_1}\log(n+1))^{\gamma_2}\sum_{j=0}^{k-1}\theta^{\frac{j}{\gamma_1+1}} \le cn^{-\gamma_1}\log(n+1))^{\gamma_2}. \quad (89)\end{aligned}$$

Combining (86–89), we conclude

$$\begin{aligned}&\left(\mathbb{E}\,\|u-u_{ik}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\ &\quad\le\ cm^{-1}n^{-\gamma_1}(\log(n+1))^{\gamma_2}+(1+2\theta)\left(\mathbb{E}\,\|u(\,\cdot\,,t_i)-u_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2}. \quad (90)\end{aligned}$$

In particular, taking into account (64), (78), and (81), we obtain by recursion over i,

$$\begin{aligned}&\left(\mathbb{E}\,\|u(\,\cdot\,,t_{i+1})-u_{i+1,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2}\\ &\le\ cm^{-1}n^{-\gamma_1}(\log(n+1))^{\gamma_2}\sum_{l=0}^{i}(1+2\theta)^l+(1+2\theta)^{i+1}\|u(\,\cdot\,,0)-u_{0,0}\|_{C(Q,\mathbb{R}^q)}\\ &\le\ c(1+2\theta)^m n^{-\gamma_1}(\log(n+1))^{\gamma_2}\le cn^{-\gamma_1}(\log(n+1))^{\gamma_2}.\end{aligned}$$

Inserting this into (90), we get

$$\left(\mathbb{E}\,\|u-u_{ik}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\le cn^{-\gamma_1}(\log(n+1))^{\gamma_2}$$

and hence,

$$\begin{aligned}&\left(\mathbb{E}\,\|u-A^{2,r}_{\nu(n),\omega}(f,u_0)\|^2_{C(Q\times[0,1],\mathbb{R}^q)}\right)^{1/2}\\ &=\ \left(\mathbb{E}\,\|u-u_{\nu(n)}\|^2_{C(Q\times[0,1],\mathbb{R}^q)}\right)^{1/2}\le cmn^{-\gamma_1}(\log(n+1))^{\gamma_2}\le cn^{-\gamma_1}(\log(n+1))^{\gamma_2},\end{aligned}$$

which is (71).

Finally we prove the stability (72) of algorithm $A^{2,r}_{\nu(n),\omega}$. Let $(f,u_0),(\tilde f,\tilde u_0)\in F_2^0(\kappa)$, let $\tilde u_{ij}$ and $\tilde g_{ij}$ be defined analogously to (62–64) and set

$$g^*_{ij}(s,t)=f(s,t,\tilde u_{ij}(s,t))\qquad(s\in Q,t\in[t_i,t_{i+1}]). \quad (91)$$

From (62) we get

$$\begin{aligned}
&\left(\mathbb{E}\,\|u_{i,j+1}-\tilde{u}_{i,j+1}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\le \left(\mathbb{E}\,\|u_{i,0}-\tilde{u}_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2}\\
&\quad+\left(\mathbb{E}\,\left\|A^{1,r}_{\mu(n_j),i,\omega}g_{ij}-A^{1,r}_{\mu(n_j),i,\omega}g^*_{ij}\right\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\quad+\left(\mathbb{E}\,\left\|A^{1,r}_{\mu(n_j),i,\omega}g^*_{ij}-A^{1,r}_{\mu(n_j),i,\omega}\tilde{g}_{ij}\right\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}. \qquad (92)
\end{aligned}$$

We have by (82) and (91)

$$g_{ij}(s,t)-g^*_{ij}(s,t) \;=\; f(s,t,u_{ij}(s,t))-f(s,t,\tilde{u}_{ij}(s,t)),$$

hence

$$\|g_{ij}-g^*_{ij}\|_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\le\kappa\|u_{ij}-\tilde{u}_{ij}\|_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}.$$

It follows from (77) that

$$\begin{aligned}
&\left(\mathbb{E}\,\left\|A^{1,r}_{\mu(n_j),i,\omega}g_{ij}-A^{1,r}_{\mu(n_j),i,\omega}g^*_{ij}\right\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&= \left(\mathbb{E}\,\mathbb{E}\left(\left\|A^{1,r}_{\mu(n_j),i,\omega}g_{ij}-A^{1,r}_{\mu(n_j),i,\omega}g^*_{ij}\right\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\,\Big|(u_{ij},\tilde{u}_{ij})\right)\right)^{1/2}\\
&\le (c(1)+1)m^{-1}\left(\mathbb{E}\,\|g_{ij}-g^*_{ij}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\le (c(1)+1)m^{-1}\kappa\left(\mathbb{E}\,\|u_{ij}-\tilde{u}_{ij}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}. \qquad (93)
\end{aligned}$$

Similarly,

$$g^*_{ij}(s,t)-\tilde{g}_{ij}(s,t) \;=\; f(s,t,\tilde{u}_{ij}(s,t))-\tilde{f}(s,t,\tilde{u}_{ij}(s,t)),$$

which yields

$$\|g^*_{ij}-\tilde{g}_{ij}\|_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\le\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)}.$$

Using again (77), we conclude

$$\begin{aligned}
&\left(\mathbb{E}\,\left\|A^{1,r}_{\mu(n_j),i,\omega}g^*_{ij}-A^{1,r}_{\mu(n_j),i,\omega}\tilde{g}_{ij}\right\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&= \left(\mathbb{E}\,\mathbb{E}\left(\left\|A^{1,r}_{\mu(n_j),i,\omega}g^*_{ij}-A^{1,r}_{\mu(n_j),i,\omega}\tilde{g}_{ij}\right\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\,\Big|\tilde{u}_{ij}\right)\right)^{1/2}\\
&\le (c(1)+1)m^{-1}\left(\mathbb{E}\,\|g^*_{ij}-\tilde{g}_{ij}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}\\
&\le (c(1)+1)m^{-1}\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)}. \qquad (94)
\end{aligned}$$

Combining (78), and (92–94), we obtain

$$
\begin{aligned}
&\left(\mathbb{E}\,\|u_{i,j+1}-\tilde{u}_{i,j+1}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2} \\
&\le \left(\mathbb{E}\,\|u_{i,0}-\tilde{u}_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2} + (c(1)+1)m^{-1}\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)} \\
&\quad +\theta\left(\mathbb{E}\,\|u_{ij}-\tilde{u}_{ij}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2}.
\end{aligned}
$$

Recursion over j together with (88) gives

$$
\begin{aligned}
&\left(\mathbb{E}\,\|u_{ik}-\tilde{u}_{ik}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2} \\
&\le \left(\left(\mathbb{E}\,\|u_{i,0}-\tilde{u}_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2} + (c(1)+1)m^{-1}\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)}\right) \\
&\quad \times(1+\theta+\dots+\theta^{k-1}) + \theta^k\left(\mathbb{E}\,\|u_{i,0}-\tilde{u}_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2} \\
&\le (1+2\theta)(c(1)+1)m^{-1}\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)} \\
&\quad +(1+2\theta)\left(\mathbb{E}\,\|u_{i,0}-\tilde{u}_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2}. \qquad (95)
\end{aligned}
$$

Consequently, recalling (64) and using recursion over i, we obtain

$$
\begin{aligned}
&\left(\mathbb{E}\,\|u_{i+1,0}-\tilde{u}_{i+1,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2} \\
&\le (1+2\theta)(c(1)+1)m^{-1}\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)} \\
&\quad +(1+2\theta)\left(\mathbb{E}\,\|u_{i,0}-\tilde{u}_{i,0}\|^2_{C(Q,\mathbb{R}^q)}\right)^{1/2} \\
&\le (1+2\theta+(1+2\theta)^2+\dots+(1+2\theta)^{i+1})(c(1)+1)m^{-1} \\
&\quad \times\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)} + (1+2\theta)^{i+1}\|u_0-\tilde{u}_0\|_{C(Q,\mathbb{R}^q)} \\
&\le c(\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)} + \|u_0-\tilde{u}_0\|_{C(Q,\mathbb{R}^q)}).
\end{aligned}
$$

Combining this with (95) yields

$$
\left(\mathbb{E}\,\|u_{ik}-\tilde{u}_{ik}\|^2_{C(Q\times[t_i,t_{i+1}],\mathbb{R}^q)}\right)^{1/2} \le c(\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)} + \|u_0-\tilde{u}_0\|_{C(Q,\mathbb{R}^q)}),
$$

and finally

$$
\begin{aligned}
&\left(\mathbb{E}\,\|A^{2,r}_{\nu(n),\omega}(f,u_0)-A^{2,r}_{\nu(n),\omega}(\tilde{f},\tilde{u}_0)\|^2_{C(Q\times[0,1],\mathbb{R}^q)}\right)^{1/2} \\
&\le c(\|f-\tilde{f}\|_{C(Q\times[0,1]\times\mathbb{R}^q,\mathbb{R}^q)} + \|u_0-\tilde{u}_0\|_{C(Q,\mathbb{R}^q)}).
\end{aligned}
$$

□

Now we will work in the setting of information-based complexity theory (IBC), see [24, 22]. For the precise notions used here we also refer to [10, 11]. An abstract numerical problem is described by a tuple (F, G, S, K, Λ), with F an arbbitrary set – the set of input data, G a normed linear space and $S : F \to G$ an arbitrary mapping, the solution operator, which maps the input $f \in F$ to the exact solution Sf. Furthermore, K is an arbitrary set and Λ is a set of mappings from F to K – the class of admissible information functionals.

The cardinality of an algorithm A, denoted by $\operatorname{card}(A)$, is the number of information functionals used in A. Let $e_n^{\rm det}(S, F, G)$, respectively $e_n^{\rm ran}(S, F, G)$, denote the n-th minimal error in the deterministic, respectively randomized setting, that is, the minimal possible error among all deterministic, respectively randomized algorithms of cardinality at most n. The cardinality of an algorithm A is closely related to the arithmetic cost, that is, the number of arithmetic operations needed to carry out A. For many concrete algorithms, including all those considered here, the arithmetic cost is within a constant or a logarithmic factor of $\operatorname{card}(A)$.

To put the parametric ODE problem into the setting above, let

$$S = S_2, \quad F = F_2^r(\kappa), \quad G = C(Q \times [0, 1], \mathbb{R}^q), \quad K = \mathbb{R}^q,$$

and let Λ_2 be the following class of function values

$$\Lambda_2 = \{\delta_{s,t,z} : \ s \in Q, t \in [0, 1], z \in \mathbb{R}^q\} \cup \{\delta_s : \ s \in Q\},$$

where $\delta_{s,t,z}(f, u_0) = f(s, t, z)$ and $\delta_s(f, u_0) = u_0(s)$.

The following theorem extends a result on the complexity of parametric ODEs from [4]. There the Lipschitz condition was imposed on f and on certain derivatives of f up to order r, here the Lipschitz condition is required for f alone. This is also of importance for the applications to PDEs in the next section.

Theorem 1. *Let $r \in \mathbb{N}_0$, $d, q \in \mathbb{N}$, $\kappa > 0$. Then the deterministic n-th minimal errors satisfy*

$$e_n^{\rm det}(S_2, F_2^r(\kappa), C(Q \times [0, 1], \mathbb{R}^q)) \asymp n^{-\frac{r}{d+1}}.$$

For the randomized n-th minimal errors we have the following: If $r/d > 1/2$, then

$$e_n^{\rm ran}(S_2, F_2^r(\kappa), C(Q \times [0, 1], \mathbb{R}^q)) \asymp n^{-\frac{r+1/2}{d+1}} (\log n)^{\frac{1}{2}},$$

if $r/d = 1/2$, then

$$n^{-\frac{1}{2}} (\log n)^{\frac{1}{2}} \preceq e_n^{\rm ran}(S_2, F_2^r(\kappa), C(Q \times [0, 1], \mathbb{R}^q)) \preceq n^{-\frac{1}{2}} (\log n)^2,$$

and if $r/d < 1/2$, then

$$e_n^{\rm ran}(S_2, F_2^r(\kappa), C(Q \times [0, 1], \mathbb{R}^q)) \asymp n^{-\frac{r}{d}} (\log n)^{\frac{r}{d}}.$$

Proof. Proposition 4 gives the upper bounds. To prove the lower bounds, we let $u_0 \equiv 0$ and consider functions $f = f(s, t)$ not depending on z. In this sense we have $\kappa B_{C^r(Q\times[0,1],\mathbb{R}^q)} \subset F_2^r(\kappa)$ and for $f \in \kappa B_{C^r(Q\times[0,1],\mathbb{R}^q)}$

$$(S_2(0, f))(s, 1) = \int_0^1 f(s, t) dt \quad (s \in Q).$$

This means that parametric definite integration of $C^r(Q \times [0,1], \mathbb{R}^q)$ functions reduces to S_2, so that the required lower bounds for parametric ODEs follow from [14].

$\square$

6 Almost Linear First Order PDEs

Let $d, r \in \mathbb{N}$ (note that throughout this section we assume $r \geq 1$), $Q = [0,1]^d$, and $\kappa > 0$. Given

$$(f, g, u_0) \in F_3^r(\kappa) := \kappa B_{C^r([0,1]\times\mathbb{R}^d,\mathbb{R}^d)} \times \kappa B_{C^r([0,1]\times\mathbb{R}^d\times\mathbb{R})} \times \kappa B_{C^r(\mathbb{R}^d)}, \tag{96}$$

$f = (f_1, \dots, f_d)$, we consider the following scalar first order almost linear PDE

$$\frac{\partial u(t,x)}{\partial t} + \sum_{i=1}^d f_i(t,x)\frac{\partial u(t,x)}{\partial x_i} = g(t,x,u(t,x)) \quad (x \in \mathbb{R}^d, t \in [0,1]), \tag{97}$$

$$u(0,x) = u_0(x). \tag{98}$$

A solution is a continuously differentiable function $u : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ satisfying (97–98). Due to the definition of $F_3^r(\kappa)$, the solution exists and is unique, see, e.g., [23], as well as the discussion of the relations to ODEs below. We seek to determine the solution at time $t = 1$ on Q, thus, we set $G_3 = C(Q)$ and define the solution operator by

$$S_3 : F_3^r(\kappa) \to C(Q), \quad (S_3(f,g,u_0))(x) = u(1,x) \quad (x \in Q).$$

Furthermore, we put $K = \mathbb{R}^d \cup \mathbb{R}$ and let Λ_3 be the following class of function values

$$\Lambda_3 = \{\delta_{t,x} : \ t \in [0,1], x \in \mathbb{R}^d\} \cup \{\delta_{t,x,z} : \ t \in [0,1], x \in \mathbb{R}^d, z \in \mathbb{R}\} \cup \{\delta_x : \ x \in Q\},$$

where

$$\delta_{t,x}(f,g,u_0) = f(t,x), \quad \delta_{t,x,z}(f,g,u_0) = g(t,x,z), \quad \delta_x(f,g,u_0) = u_0(x).$$

We use the method of characteristics. We want to find $\xi : Q \times [0,1] \to \mathbb{R}^d$ such that for $s \in Q$, $t \in [0,1]$,

$$\frac{\partial \xi(s,t)}{\partial t} = f(t, \xi(s,t)) \tag{99}$$

$$\xi(s,1) = s. \tag{100}$$

Observe that, due to (96) and the assumption $r \geq 1$,

$$\|f\|_{C^r_{\mathrm{Lip}}(Q\times[0,1]\times\mathbb{R}^d,\mathbb{R}^d)} \leq \sqrt{d}\kappa \tag{101}$$

(in the sense that $f = f(t, z)$ is considered as a function not depending on $s \in Q$). Thus, the solution of (99–100) exists and is unique. Denote

$$\xi_0 : Q \to \mathbb{R}^d, \quad \xi_0(s) = s \quad (s \in Q).$$

Let

$$\tilde{S}_2 : C_{\text{Lip}}(Q \times [0,1] \times \mathbb{R}^d, \mathbb{R}^d) \times C(Q, \mathbb{R}^d) \to C(Q \times [0,1], \mathbb{R}^d)$$

be the solution operator of parametric ODEs studied in Section 5, with the difference that the starting time is $t = 1$ and the ODE is considered backward in time (clearly, this does not affect the error estimates of Proposition 5, provided the algorithms are modified in the corresponding way). So we have

$$\xi = \tilde{S}_2(f, \xi_0).$$

Furthermore, $\|\xi_0\|_{C^r(Q,\mathbb{R}^d)} = \sqrt{d}$, and consequently, by (55) and (101), there is a $\kappa_1 > 0$ depending only on r, d and κ such that

$$\|\xi\|_{C^r(Q\times[0,1],\mathbb{R}^d)} \le \kappa_1. \tag{102}$$

We define $h \in C_{\text{Lip}}(Q \times [0,1] \times \mathbb{R})$ and $w_0 \in C(Q)$ by setting

$$h(s,t,z) = g(t, \xi(s,t), z) \tag{103}$$
$$w_0(s) = u_0(\xi(s,0)) \tag{104}$$

for $s \in Q$, $t \in [0,1]$, $z \in \mathbb{R}$. By (96) and (102), there is a $\kappa_2 > 0$ also depending only on r, d and κ such that

$$(h, w_0) \in F_2^r(\kappa_2) \subseteq F_2^0(\kappa_2). \tag{105}$$

Next we seek to find $w : Q \times [0,1] \to \mathbb{R}$ with

$$\begin{aligned} \frac{\partial w(s,t)}{\partial t} &= h(s,t,w(s,t)) \quad (s \in Q, t \in [0,1]) \\ w(s,0) &= w_0(s) \quad (s \in Q). \end{aligned}$$

Then we have

$$w = S_2(h, w_0), \tag{106}$$

where

$$S_2 : C_{\text{Lip}}(Q \times [0,1] \times \mathbb{R}) \times C(Q) \to C(Q \times [0,1])$$

is the respective solution operator of parametric ODEs, here with $q = 1$ and starting time $t = 0$. The following is well-known (see again, e.g., [23]).

Lemma 1. *If $u(t,x)$ is the solution of (97-98), then*

$$u(t, \xi(s,t)) = w(s,t) \quad (s \in Q, t \in [0,1]). \tag{107}$$

It follows from (100) and (107) that

$$u(1,s) = w(s,1) \quad (s \in Q),$$

hence by (106)

$$S_3(f,g,u_0) = (S_2(h,w_0))\,(\,\cdot\,,1). \tag{108}$$

Now let $\sigma = (\tilde{\nu},\nu) \in \mathscr{N}^2$, where $\mathscr{N}$ was defined in (60), and let $\tilde{A}^{2,r}_{\tilde{\nu},\omega}$ be the algorithm (62–66) for $\tilde{S}_2$. Similarly, let $A^{2,r}_{\nu,\omega}$ be the respective algorithm for S_2. We assume that the random variables $\tilde{A}^{2,r}_{\tilde{\nu},\omega}$ and $A^{2,r}_{\nu,\omega}$ are independent. Define $\xi_{\tilde{\nu}} \in C(Q \times [0,1], \mathbb{R}^d)$ by

$$\xi_{\tilde{\nu}} \;=\; \tilde{A}^{2,r}_{\tilde{\nu},\omega}(f,\xi_0) \tag{109}$$

and $h_{\tilde{\nu}} \in C_{\mathrm{Lip}}(Q \times [0,1] \times \mathbb{R})$, $w_{0,\tilde{\nu}} \in C(Q)$ by setting for $s \in Q$, $t \in [0,1]$, $z \in \mathbb{R}$

$$h_{\tilde{\nu}}(s,t,z) \;=\; g(t,\xi_{\tilde{\nu}}(s,t),z) \tag{110}$$

$$w_{0,\tilde{\nu}}(s) \;=\; u_0(\xi_{\tilde{\nu}}(s,0)). \tag{111}$$

It follows from (96) that

$$(h_{\tilde{\nu}}, w_{0,\tilde{\nu}}) \in F_2^0(\kappa). \tag{112}$$

We define algorithm $A^{3,r}_{\sigma,\omega}$ for S_3 by setting for $s \in Q$

$$\left(A^{3,r}_{\sigma,\omega}(f,g,u_0)\right)(s) = \left(A^{2,r}_{\nu,\omega}(h_{\tilde{\nu}}, w_{0,\tilde{\nu}})\right)(s,1). \tag{113}$$

We have $A^{3,r}_{\sigma,\omega}(f,g,u_0) \in C(Q)$. The following result provides error estimates for $A^{3,r}_{\sigma,\omega}$ (recall also the definition (61) of $\mathscr{N}_0$).

Proposition 5. *Let $r,d \in \mathbb{N}$, $\kappa > 0$. There are constants $c_{1-4} > 0$ such that the following hold. For each $n \in \mathbb{N}$ there is a $\sigma_0(n) \in \mathscr{N}_0^2$ such that for all $\omega \in \Omega$.*

$$\operatorname{card}\left(A^{3,r}_{\sigma_0(n),\omega}\right) \;\le\; c_1 n \tag{114}$$

$$\sup_{(f,g,u_0)\in F_3^r(\kappa)} \|S_3(f,g,u_0) - A^{3,r}_{\sigma_0(n),\omega}(f,g,u_0)\|_{C(Q)} \;\le\; c_2 n^{-\frac{r}{d+1}}. \tag{115}$$

Moreover, for each $n \in \mathbb{N}$ there is a $\sigma(n) \in \mathscr{N}^2$ such that

$$\sup_{\omega\in\Omega} \operatorname{card}\left(A^{3,r}_{\sigma(n),\omega}\right) \;\le\; c_3 n \tag{116}$$

$$\sup_{(f,g,u_0)\in F_3^r(\kappa)} \left(\mathbb{E}\,\|S_3(f,g,u_0) - A^{3,r}_{\sigma(n),\omega}(f,g,u_0)\|^2_{C(Q)}\right)^{1/2} \;\le\; c_4 n^{-\gamma_1}(\log(n+1))^{\gamma_2}, \tag{117}$$

with γ_1, γ_2 given by (34).

Proof. Again we only prove the stochastic case (116–117), the deterministic case being analogous. Let $(\tilde\nu(n))_{n=1}^\infty$ be such that (70–72) of Proposition 4 hold for $\tilde S_2$. Similarly, let $(\nu(n))_{n=1}^\infty$ be a respective sequence for S_2. We put $\sigma(n) = (\tilde\nu(n), \nu(n))$. Now let $n \in \mathbb{N}$, $(f, g, u_0) \in F_3^r(\kappa)$. By (71–72) of Proposition 4, (105), (112), (113), and (108)

$$
\begin{aligned}
&\left(\mathbb{E}\left\|S_3(f,g,u_0) - A^{3,r}_{\sigma(n),\omega}(f,g,u_0)\right\|^2_{C(Q)}\right)^{1/2}\\
&= \left(\mathbb{E}\left\|(S_2(h,w_0))(\,\cdot\,,1) - \left(A^{2,r}_{\nu(n),\omega}(h_{\tilde\nu(n)}, w_{0,\tilde\nu(n)})\right)(\,\cdot\,,1)\right\|^2_{C(Q)}\right)^{1/2}\\
&\le \left(\mathbb{E}\left\|S_2(h,w_0) - A^{2,r}_{\nu(n),\omega}(h_{\tilde\nu(n)}, w_{0,\tilde\nu(n)})\right\|^2_{C(Q\times[0,1])}\right)^{1/2}\\
&\le \left(\mathbb{E}\left\|S_2(h,w_0) - A^{2,r}_{\nu(n),\omega}(h,w_0)\right\|^2_{C(Q\times[0,1])}\right)^{1/2}\\
&\quad + \left(\mathbb{E}\,\mathbb{E}\left\|A^{2,r}_{\nu(n),\omega}(h,w_0) - A^{2,r}_{\nu(n),\omega}(h_{\tilde\nu(n)}, w_{0,\tilde\nu(n)})\right\|^2_{C(Q\times[0,1])}\Big|\xi_{\tilde\nu(n)}\right)^{1/2}\\
&\le cn^{-\gamma_1}(\log(n+1))^{\gamma_2} + c\left(\mathbb{E}\,\|h - h_{\tilde\nu(n)}\|^2_{C(Q\times[0,1]\times\mathbb{R})}\right)^{1/2}\\
&\quad + c\left(\mathbb{E}\,\|w_0 - w_{0,\tilde\nu(n)}\|^2_{C(Q)}\right)^{1/2}. \qquad (118)
\end{aligned}
$$

By (103), (110), and (96), for $s \in Q$, $t \in [0,1]$, $z \in \mathbb{R}$

$$
\begin{aligned}
|h(s,t,z) - h_{\tilde\nu(n)}(s,t,z)| &= |g(t,\xi(s,t),z) - g(t,\xi_{\tilde\nu(n)}(s,t),z)|\\
&\le \sqrt{d}\kappa\|\xi(s,t) - \xi_{\tilde\nu(n)}(s,t)\|_{\mathbb{R}^d}, \qquad (119)
\end{aligned}
$$

and similarly, by (104), (111), and (96),

$$
\begin{aligned}
|w_0(s) - w_{0,\tilde\nu(n)}(s)| &= |u_0(\xi(s,0)) - u_0(\xi_{\tilde\nu(n)}(s,0))|\\
&\le \sqrt{d}\kappa\|\xi(s,0) - \xi_{\tilde\nu(n)}(s,0)\|_{\mathbb{R}^d}. \qquad (120)
\end{aligned}
$$

Furthermore, using (101) and (71) of Proposition 4, we obtain

$$
\begin{aligned}
&\left(\mathbb{E}\,\|\xi - \xi_{\tilde\nu(n)}\|^2_{C(Q\times[0,1],\mathbb{R}^d)}\right)^{1/2}\\
&= \left(\mathbb{E}\left\|\tilde S_2(f,\xi_0) - \tilde A^{2,r}_{\tilde\nu(n),\omega}(f,\xi_0)\right\|^2_{C(Q\times[0,1],\mathbb{R}^d)}\right)^{1/2}\\
&\le cn^{-\gamma_1}(\log(n+1))^{\gamma_2}. \qquad (121)
\end{aligned}
$$

From (119–121) we conclude

$$
\begin{aligned}
\left(\mathbb{E}\,\|h - h_{\tilde\nu(n)}\|^2_{C(Q\times[0,1]\times\mathbb{R})}\right)^{1/2} &\le cn^{-\gamma_1}(\log(n+1))^{\gamma_2} \qquad (122)\\
\left(\mathbb{E}\,\|w_0 - w_{0,\tilde\nu(n)}\|^2_{C(Q)}\right)^{1/2} &\le cn^{-\gamma_1}(\log(n+1))^{\gamma_2}. \qquad (123)
\end{aligned}
$$

Combining (118) and (122–123), we obtain the desired result (117). Relation (116) follows from the definition of $A^{3,r}_{\sigma(n),\omega}$ and (70) of Proposition 4.

□

The following theorem gives the deterministic and randomized minimal errors of the first order almost linear PDE problem.

Theorem 2. *Let $r, d \in \mathbb{N}$ and $\kappa > 0$. Then in the deterministic setting,*

$$e_n^{\det}(S_3, F_3^r(\kappa), C(Q)) \asymp n^{-\frac{r}{d+1}}.$$

In the randomized setting, if $r/d > 1/2$

$$e_n^{\mathrm{ran}}(S_3, F_3^r(\kappa), C(Q)) \asymp n^{-\frac{r+1/2}{d+1}}(\log n)^{\frac{1}{2}},$$

if $r/d = 1/2$, then

$$n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} \preceq e_n^{\mathrm{ran}}(S_3, F_3^r(\kappa), C(Q)) \preceq n^{-\frac{1}{2}}(\log n)^2,$$

and if $r/d < 1/2$, then

$$e_n^{\mathrm{ran}}(S_3, F_3^r(\kappa), C(Q)) \asymp n^{-\frac{r}{d}}(\log n)^{\frac{r}{d}}.$$

Proof. The upper bounds follow from Proposition 5 above. To show the lower bounds, we set $f \equiv 0$, $u_0 \equiv 0$, and consider $g = g(t, x)$ not depending on z. Let $C_Q^r([0, 1] \times \mathbb{R}^d)$ be the subspace of $C^r([0, 1] \times \mathbb{R}^d)$ consisting of all functions g satisfying

$$\operatorname{supp} g(t, \cdot) \subseteq Q \quad (t \in [0, 1]).$$

Then $g \in \kappa B_{C_Q^r([0,1]\times\mathbb{R}^d)}$ implies $(0, g, 0) \in F_3^r(\kappa)$. Moreover,

$$(S_3(0, g, 0))(x) = \int_0^1 g(t, x)dt \quad (x \in Q),$$

thus parametric definite integration of $C_Q^r([0, 1] \times \mathbb{R}^d)$ functions reduces to S_3, and the lower bounds follow from [14] (it is readily seen from the proof in [14] that the lower bound also holds for the subclass of functions with support in Q).

□

Note that to obtain this result it was crucial to have Proposition 4 and Theorem 1 for parametric ODEs under the Lipschitz condition as imposed in definitions (50), (51), and (54). If we wanted to apply the results of [4] to get the upper bounds as stated in Theorem 2, we would have to ensure the stronger Lipschitz condition from [4] (involving derivatives up to order r). This would mean to require $(f, g, u_0) \in F_3^{r+1}(\kappa)$, which, in turn, would lead to gaps between the upper and lower bounds in Theorem 2 (in the lower bounds r would have to be replaced by $r + 1$).

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