

# Complexity of stochastic integration in Sobolev classes

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## Abstract

We study the complexity of stochastic integration with respect to an isonormal process defined on a bounded Lipschitz domain  $Q \subset \mathbb{R}^d$ . We consider integration of functions from Sobolev spaces  $W_p^r(Q)$  and analyze the complexity in the deterministic and randomized setting. Matching upper and lower bounds for the  $n$ -th minimal error are established, this way determining the complexity of the problem. It turns out that the stochastic integration problem is closely related to approximation of the embedding of  $W_p^r(Q)$  into  $L_2(Q)$ .

## 1 Introduction

Let  $Q \subset \mathbb{R}^d$  be a bounded Lipschitz domain and let  $W_Q$  be an isonormal process on  $L_2(Q)$ . We study the complexity of pathwise approximation of the stochastic integral  $\int_Q f(x)dW_Q(x)$ . Here  $f$  is a function from some Sobolev space  $W_p^r(Q)$  (embedded in  $L_2(Q)$ ). We determine the complexity in various settings.

On the way to this main result we first review the general approach of information-based complexity theory (IBC) to stochastic problems, introduced in [7]. We complement it by considering besides adaptive deterministic and randomized algorithms a third setting – that of semi-adaptive algorithms, which turns out to be much closer to the deterministic setting for deterministic problems than the larger class of adaptive deterministic algorithms. We prove some new general results, in particular by exploiting the connection to deterministic problems in both the deterministic and randomized settings. It turns out that the stochastic integration problem is closely related to approximation of the embedding of the respective function space into  $L_2(Q)$ . Furthermore, lower bounds for stochastic integration in general function classes on  $Q$  are proved. Finally, the complexity for Sobolev classes is determined in all three settings. We also treat the case of Slobodeckij spaces. Algorithms and error estimates for some of the latter were obtained by Eisenmann and Kruse [3]. Our results provide matching lower bounds. Our methods are extensions of those from [7] and [6].

The complexity of stochastic integration of functions on finite intervals in  $\mathbb{R}$  was investigated in [22], [8], [15], [16]. We also mention related work on the complexity of stochastic differential equations [9, 10]. In the present paper the complexity of stochastic integration with respect to a  $d$ -dimensionally indexed stochastic process is considered for the first time. In particular, for  $Q = [0, 1]^d$ , our study includes the integral with respect to the Wiener sheet measure.

The paper is organized as follows. In Section 2 we review and enlarge the general theory, in Section 3 the needed previous results for deterministic problems in the randomized setting are stated. Section 4 contains new results on the general theory, while stochastic integration for arbitrary subsets of  $L_2(Q)$  is studied in Section 5. The final Section 6 contains the main result about Sobolev classes and an analogous statement for Slobodeckij spaces.

Let  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ . The unit ball of a Banach space  $X$  is denoted by  $B_X$ . We often use the same symbol  $c, c_1, \dots$  for possibly different positive constants (also when they appear in a sequence of relations). For sequences  $a_n, b_n \geq 0$  ( $n \in \mathbb{N}$ ) we write  $a_n \preceq b_n$  if there are constants  $n_0 \in \mathbb{N}$  and  $c > 0$  such that  $a_n \leq cb_n$  for all  $n \geq n_0$ . Moreover,  $a_n \succeq b_n$  stands for  $b_n \preceq a_n$ , while  $a_n \asymp b_n$  means that both  $a_n \preceq b_n$  and  $b_n \preceq a_n$  hold.

## 2 Preliminaries: A general setting of IBC for stochastic problems

In this section we recall the IBC approach to stochastic problems from [7]. An abstract stochastic numerical problem  $\mathcal{P}$  is given as

$$\mathcal{P} = (F, (\Omega, \Sigma, \mathbb{P}), G, S, K, \Lambda). \quad (1)$$

Here  $F$  is a non-empty set,  $(\Omega, \Sigma, \mathbb{P})$  a probability space,  $G$  a Banach space and  $S$  is a mapping  $F \times \Omega \rightarrow G$ . The operator  $S$  is called the solution operator, it sends the input  $(f, \omega) \in F \times \Omega$  of our problem to the exact solution  $S(f, \omega)$ . Moreover,  $\Lambda$  is a nonempty set of mappings from  $F \times \Omega$  to  $K$ , the set of information functionals, where  $K$  is any nonempty set - the set of values of information functionals.

We assume that for each  $f \in F$  the mapping  $\omega \rightarrow S(f, \omega)$  is  $\Sigma$ -to-Borel-measurable and  $\mathbb{P}$ -almost surely separably valued, i.e., for each  $f \in F$  there is a separable subspace  $G_f$  of  $G$  such that  $\mathbb{P}\{\omega : S(f, \omega) \in G_f\} = 1$ .

An (adaptive) deterministic algorithm for  $\mathcal{P}$  is a tuple  $A = ((L_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty)$  such that  $L_1 \in \Lambda$ ,  $\tau_0 \in \{0, 1\}$ ,  $\varphi_0 \in G$ , and for  $i \in \mathbb{N}$

$$L_{i+1} : K^i \rightarrow \Lambda, \quad \tau_i : K^i \rightarrow \{0, 1\}, \quad \varphi_i : K^i \rightarrow G \quad (2)$$

are arbitrary mappings. Given an input  $(f, \omega) \in F \times \Omega$ , we define  $(\lambda_i)_{i=1}^\infty$  with  $\lambda_i \in \Lambda$  as follows:

$$\lambda_1 = L_1, \quad \lambda_i = L_i(\lambda_1(f, \omega), \dots, \lambda_{i-1}(f, \omega)) \quad (i \geq 2). \quad (3)$$

Define  $\text{card}(A, f, \omega)$ , the cardinality of  $A$  at input  $(f, \omega)$ , to be 0 if  $\tau_0 = 1$ . If  $\tau_0 = 0$ , let  $\text{card}(A, f, \omega)$  be the first integer  $n \geq 1$  with  $\tau_n(\lambda_1(f, \omega), \dots, \lambda_n(f, \omega)) = 1$  if there is such an  $n$ . If  $\tau_0 = 0$  and no such  $n \in \mathbb{N}$  exists, put  $\text{card}(A, f, \omega) = +\infty$ . Observe that we have the following alternative: Either

$$\text{card}(A, f, \omega) = 0 \text{ for all } (f, \omega) \in F \times \Omega \quad \text{or} \quad \text{card}(A, f, \omega) \geq 1 \text{ for all } (f, \omega) \in F \times \Omega. \quad (4)$$

We define the output  $A(f, \omega)$  of algorithm  $A$  at input  $(f, \omega)$  as

$$A(f, \omega) = \begin{cases} \varphi_0 & \text{if } \text{card}(A, f, \omega) \in \{0, \infty\} \\ \varphi_n(\lambda_1(f, \omega), \dots, \lambda_n(f, \omega)) & \text{if } 1 \leq \text{card}(A, f, \omega) = n < \infty. \end{cases} \quad (5)$$

Define  $k^* = K$  and  $K^0 = \{k^*\}$ . (This is a technical definition which guarantees that  $K^0$  is a one-element set whose element does not belong to any  $K^i$  for  $i \geq 1$ .) Let  $K^\infty = \cup_{i=0}^\infty K^i$  and define a mapping, the information operator,  $N : F \times \Omega \rightarrow K^\infty$  as

$$N(f, \omega) = \begin{cases} k^* \in K^0 & \text{if } \text{card}(A, f, \omega) \in \{0, \infty\} \\ (\lambda_1(f, \omega), \dots, \lambda_n(f, \omega)) \in K^n & \text{if } 1 \leq \text{card}(A, f, \omega) = n < \infty. \end{cases} \quad (6)$$

Furthermore, define a mapping  $\varphi : K^\infty \rightarrow G$  by setting for  $a \in K^\infty$

$$\varphi(a) = \begin{cases} \varphi_0 & \text{if } a = k^* \\ \varphi_n(a_1, \dots, a_n) & \text{if } a = (a_1, \dots, a_n) \in K^n, n \in \mathbb{N}. \end{cases} \quad (7)$$

This gives a convenient representation  $A = \varphi \circ N$ , that is,

$$A(f, \omega) = \varphi(N(f, \omega)) \quad ((f, \omega) \in F \times \Omega). \quad (8)$$

Given  $n \in \mathbb{N}_0$ , we define  $\mathcal{A}_n^{\text{det}}(\mathcal{P})$  as the set of those deterministic algorithms  $A$  for  $\mathcal{P}$  with the following properties: For each  $f \in F$  the mapping  $\omega \rightarrow \text{card}(A, f, \omega)$  is  $\Sigma$ -measurable,  $\mathbb{E} \text{card}(A, f, \omega) \leq n$ , and the mapping  $\omega \rightarrow A(f, \omega) \in G$  is  $\Sigma$ -to-Borel-measurable and  $\mathbb{P}$ -almost surely separably valued. The cardinality of  $A \in \mathcal{A}_n^{\text{det}}(\mathcal{P})$  is defined as

$$\text{card}(A, F \times \Omega) = \sup_{f \in F} \mathbb{E} \text{card}(A, f, \omega),$$

the error of  $A$  in approximating  $S$  as

$$e(S, A, F \times \Omega, G) = \sup_{f \in F} \mathbb{E} \|S(f, \omega) - A(f, \omega)\|_G$$

and the deterministic  $n$ -th minimal error of  $S$  is defined for  $n \in \mathbb{N}_0$  as

$$e_n^{\text{det}}(S, F \times \Omega, G) = \inf_{A \in \mathcal{A}_n^{\text{det}}(\mathcal{P})} e(S, A, F \times \Omega, G). \quad (9)$$

A randomized algorithm for  $\mathcal{P}$  is a tuple  $A = ((\Omega_1, \Sigma_1, \mathbb{P}_1), (A_{\omega_1})_{\omega_1 \in \Omega_1})$ , where  $(\Omega_1, \Sigma_1, \mathbb{P}_1)$  is another probability space and for each  $\omega_1 \in \Omega_1$ ,  $A_{\omega_1}$  is a deterministic algorithm for  $\mathcal{P}$ . Let  $n \in \mathbb{N}_0$ . Then  $\mathcal{A}_n^{\text{ran}}(\mathcal{P})$  stands for the class of randomized algorithms  $A$  for  $\mathcal{P}$  with the following properties: For each  $f \in F$  the mapping  $(\omega_1, \omega) \rightarrow \text{card}(A_{\omega_1}, f, \omega)$  is  $\Sigma_1 \times \Sigma$ -measurable,

$$\mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}} \text{card}(A_{\omega_1}, f, \omega) \leq n,$$

and the mapping  $(\omega_1, \omega) \rightarrow A_{\omega_1}(f, \omega)$  is  $\Sigma_1 \times \Sigma$ -to-Borel-measurable and  $\mathbb{P}_1 \times \mathbb{P}$ -almost surely separably valued. We define the cardinality of  $A \in \mathcal{A}_n^{\text{ran}}(\mathcal{P})$  as

$$\text{card}(A) = \sup_{f \in F} \mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}} \text{card}(A_{\omega_1}, f, \omega),$$

the error as

$$e(S, A, F \times \Omega, G) = \sup_{f \in F} \mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}} \|S(f, \omega) - A_{\omega_1}(f, \omega)\|_G$$

and the randomized  $n$ -th minimal error of  $S$  as

$$e_n^{\text{ran}}(S, F \times \Omega, G) = \inf_{A \in \mathcal{A}_n^{\text{ran}}(\mathcal{P})} e(S, A, F \times \Omega, G).$$

Considering trivial one-point probability spaces  $\Omega_1 = \{\omega_1\}$  immediately yields

$$e_n^{\text{ran}}(S, F \times \Omega, G) \leq e_n^{\text{det}}(S, F \times \Omega, G). \quad (10)$$

By definition (3), deterministic algorithms can use the intrinsic randomness of the stochastic problem, so the choice of information about  $f$  may depend on  $\omega$ , in general. Therefore in some concrete cases, including the stochastic integration problems considered here, deterministic algorithms can simulate randomized ones by using the additional randomness in the problem (see, e.g., the proof of Lemma 5.2). This leads to the same rates for deterministic and stochastic minimal errors. We want to single out a class of algorithms where this is not the case, that is, the information collected about  $f \in F$  does not depend on  $\omega$ . This class is much closer to the class of deterministic algorithms for deterministic problems than the class above. There are some interesting lower bounds statements connected with this class, see (50), (77), and (78).

First let us formalize this notion. For this purpose let  $\Lambda_F \subseteq \Lambda$  be the subset of all  $\lambda \in \Lambda$  which depend only on  $f \in F$ , precisely,

$$\Lambda_F = \{\lambda \in \Lambda : \lambda(f, \omega_1) = \lambda(f, \omega_2) \text{ for all } f \in F, \omega_1, \omega_2 \in \Omega\},$$

and correspondingly,  $\Lambda_\Omega \subseteq \Lambda$  the set of all those  $\lambda \in \Lambda$  which depend only on  $\omega \in \Omega$ . A deterministic algorithm  $A = ((L_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty)$  is called semi-adaptive, if the following hold. There is a sequence  $(\sigma_i)_{i=0}^\infty$  with  $\sigma_0 \in \{0, 1\}$  and  $\sigma_i : K^i \rightarrow \{0, 1\}$  ( $i \in \mathbb{N}$ ) being any mappings satisfying  $\sigma_i \geq \tau_i$ . In analogy to the cardinality above we define  $\text{card}_F(A, f, \omega)$  for  $(f, \omega) \in F \times \Omega$  to be 0 if  $\sigma_0 = 1$ , while if  $\sigma_0 = 0$ , we let  $\text{card}_F(A, f, \omega)$  be the first integer  $m \in \mathbb{N}$  with  $\sigma_m(\lambda_1(f, \omega), \dots, \lambda_m(f, \omega)) = 1$  (with the  $\lambda_i$  given by (3)), if there is such an  $m$ . If no such  $m \in \mathbb{N}$  exists, set  $\text{card}_F(A, f, \omega) = \infty$ . Then for all  $f$  and  $\omega$  the following has to be satisfied for  $i \in \mathbb{N}$ .

$$\lambda_i \in \begin{cases} \Lambda_F & \text{if } i \leq \text{card}_F(A, f, \omega) \\ \Lambda_\Omega & \text{if } i > \text{card}_F(A, f, \omega). \end{cases}$$

This concludes the definition of a semi-adaptive algorithm. First note that  $\text{card}_F(A, f, \omega)$  does not depend on  $\omega$ , so we write  $\text{card}_F(A, f)$  instead. Similarly,  $\lambda_i(f, \omega) = \lambda_i(f)$  ( $1 \leq i \leq m$ ). Also observe that we always have

$$\text{card}_F(A, f) \leq \text{card}(A, f, \omega)$$

for all  $f$  and  $\omega$ . It follows that

$$\sup_{f \in F} \text{card}_F(A, f) \leq \sup_{f \in F} \mathbb{E} \text{card}(A, f, \omega) = \text{card}(A, F). \quad (11)$$

We put

$$N_1(f) = \begin{cases} k^* \in K^0 & \text{if } \text{card}_F(A, f) = 0 \text{ or } \text{card}_F(A, f) = \infty \\ (\lambda_1(f), \dots, \lambda_m(f)) \in K^m & \text{if } 1 \leq \text{card}_F(A, f, \omega) = m < \infty. \end{cases} \quad (12)$$

So such an algorithm first collects information about  $f$ , which does not depend on  $\omega$ , but otherwise may be adaptive:  $N_1(f)$ . After that information about  $\omega$  is collected, which may depend on the previously computed values of  $f$ . Recalling (6)–(8), we note that for  $f, g \in F$  with  $N_1(f) = N_1(g)$  we have

$$N(f, \omega) = N(g, \omega), \quad \text{thus} \quad A(f, \omega) = A(g, \omega) \quad (\omega \in \Omega). \quad (13)$$

Given  $n \in \mathbb{N}_0$ , we define  $\mathcal{A}_n^{\text{dsa}}(\mathcal{P})$  as the set of those algorithms  $A \in \mathcal{A}_n^{\text{det}}(\mathcal{P})$  which are semi-adaptive. In analogy with (9) we set

$$e_n^{\text{dsa}}(S, F \times \Omega, G) = \inf_{A \in \mathcal{A}_n^{\text{dsa}}(\mathcal{P})} e(S, A, F \times \Omega, G). \quad (14)$$

Clearly, we have

$$e_n^{\text{det}}(S, F \times \Omega, G) \leq e_n^{\text{dsa}}(S, F \times \Omega, G). \quad (15)$$

The class of semi-adaptive algorithms was first considered in [16].

### 3 Lower bounds for deterministic solution operators

The abstract approach to deterministic problems can easily be obtained from that for stochastic problems described in [7] by letting  $\Omega = \{\omega_0\}$  be the trivial one-point space (in other words, all dependencies on  $\omega$  are dropped). This way we get the deterministic and randomized setting for deterministic problems [17, 12], in a form as developed in [4, 5]. This framework is used in the present section. (Observe that here the classes of deterministic and of deterministic semi-adaptive algorithms coincide.) Besides the deterministic and randomized setting we also need the average case setting. Let  $\mathcal{P} = (F, G, S, K, \Lambda)$  be a deterministic problem, let  $\nu$  be a measure on  $F$  whose support is a finite set and let  $A$  be a deterministic algorithm for  $\mathcal{P}$ . Put

$$\begin{aligned} \text{card}(A, \nu, G) &= \int_F \text{card}(A, f) d\nu(f), \\ e(S, A, \nu, G) &= \int_F \|S(f) - A(f)\|_G d\nu(f), \\ e_n^{\text{avg}}(S, \nu, G) &= \inf\{e(S, A, \nu, G) : \\ &\quad A \text{ a deterministic algorithm for } \mathcal{P} \text{ with } \text{card}(A, \nu, G) \leq n\}. \end{aligned}$$

In this section we consider linear problems under standard information, which means we assume the following. Let  $D$  be an arbitrary non-empty set,  $Z$  a linear space,  $F_1$  a linear subspace of the space of  $Z$ -valued functions on  $D$ , let  $F$  be any non-empty subset of  $F_1$ ,  $\Lambda$  any non-empty subset of  $\{\delta_x : x \in D\}$ , where  $\delta_x(f) = f(x) \in Z$ , let  $G$  be a normed space, and  $S : F \rightarrow G$  a mapping which is the restriction to  $F$  of a linear operator from  $F_1$  to  $G$ .

**Proposition 3.1.** *Let  $\bar{n}, n \in \mathbb{N}$ ,  $\bar{n} > 4n$ , and  $\{w_i\}_{i=1}^{\bar{n}} \subset F$  be such that*

$$\{x \in D : w_i(x) \neq 0\} \cap \{x \in D : w_j(x) \neq 0\} = \emptyset \quad (i \neq j). \quad (16)$$

*Then the following hold.*

1. *Assume that  $\sum_{i=1}^{\bar{n}} \beta_i w_i \in F$  for all  $\beta_i \in \{-1, 1\}$ ,  $i = 1, \dots, \bar{n}$ . Let  $(\varepsilon_i)_{i=1}^{\bar{n}}$  be independent, centered Bernoulli random variables on a probability space  $(\Omega, \Sigma, \mathbb{P})$  and define the measure  $\nu$  on  $F$  to be the distribution of  $\sum_{i=1}^{\bar{n}} \varepsilon_i w_i$ . Then*

$$e_n^{\text{avg}}(S, \nu, G) \geq \frac{1}{2} \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n} - 2n} \left( \mathbb{E} \left\| \sum_{k \in K} \varepsilon_k S w_k \right\|_G \right). \quad (17)$$

2. *Assume that  $-w_i \in F$  for  $i = 1, \dots, \bar{n}$ . Let  $\nu$  be the uniform distribution on the set*

$$\{\beta w_i : 1 \leq i \leq \bar{n}, \beta \in \{-1, 1\}\}.$$

Then

$$e_n^{\text{avg}}(S, \nu, G) \geq \frac{1}{\bar{n}} \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n} - 2n} \sum_{k \in K} \|Sw_k\|_G. \quad (18)$$

These types of lower bounds are well-known in IBC (see [12, 17]). The specific form (17) presented here can be found in [4], Lemma 6. Since (18) is less visible in the literature, we include the short proof for the sake of completeness.

*Proof.* Let  $A$  be any deterministic algorithm for  $S$  with

$$\text{card}(A, \nu) \leq n. \quad (19)$$

Let  $h_0$  denote the function which is identically 0 ( $h_0$  does not need to be in  $F$ ). Let  $q = \text{card}(A, h_0)$ , and let  $(\delta_{x_i})_{i=1}^q$  be the sequence of information functionals called by  $A$  at input  $h_0$ . We have  $0 \leq q \leq +\infty$ . First we assume  $q \geq 1$ . Define

$$I_0 = \{i : 1 \leq i \leq \bar{n}, w_i(x_j) = 0 \text{ for all } 1 \leq j \leq \min(q, 2n)\}.$$

It follows from the assumptions that

$$|I_0| \geq \bar{n} - 2n \geq \frac{\bar{n}}{2}. \quad (20)$$

We show that

$$q \leq 2n. \quad (21)$$

Assume  $q > 2n$ . Then we have

$$w_i(x_j) = 0 \quad (i \in I_0, 1 \leq j \leq 2n),$$

and, since  $\beta w_i(x_j) = h_0(x_j)$  for all  $j \leq 2n$  and  $\beta \in \{-1, 1\}$ , it follows that

$$\text{card}(A, \beta w_i) \geq 2n + 1 \quad (i \in I_0, \beta \in \{-1, 1\}).$$

Together with (20) this implies

$$\text{card}(A, \nu) \geq \frac{2|I_0|(2n+1)}{2\bar{n}} \geq \frac{2n+1}{2} > n,$$

a contradiction to (19). This proves (21), from which it follows that for all  $i \in I_0$

$$w_i(x_j) = 0 \quad (1 \leq j \leq q),$$

and hence

$$A(\beta w_i) = A(h_0) \quad (i \in I_0, \beta \in \{-1, 1\}). \quad (22)$$

Consequently, using (20),

$$e(A, \nu) \geq \frac{1}{2\bar{n}} \sum_{i \in I_0} (\|Sw_i - A(h_0)\|_G + \|-Sw_i - A(h_0)\|_G) \geq \frac{1}{\bar{n}} \sum_{i \in I_0} \|Sw_i\|_G. \quad (23)$$

Recall that so far we assumed  $q \geq 1$ . If  $q = 0$ , this means that the output  $A(f)$  does not depend on  $f$  at all. In this case (22) holds trivially, with  $I_0 = \{1, \dots, \bar{n}\}$ , and (23) follows in the same way. Since  $A$  was an arbitrary algorithm with (19), we obtain from (23)

$$e_{2n}^{\text{avg}}(S, \nu) \geq \frac{1}{\bar{n}} \sum_{i \in I_0} \|Sw_i\|_G \geq \frac{1}{\bar{n}} \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n} - 2n} \sum_{k \in K} \|Sw_k\|_G,$$

which concludes the proof.  $\square$

## 4 Lower bounds for stochastic solution operators

Let  $\mathcal{P} = (F, (\Omega, \Sigma, \mathbb{P}), G, S, K, \Lambda)$  be as in Section 2. We start with an observation about deterministic semi-adaptive algorithms. We can connect the stochastic problem  $\mathcal{P}$  with a deterministic one  $\tilde{\mathcal{P}}$ . For this purpose we assume that the original problem  $\mathcal{P}$  is such that  $S(f, \cdot) \in L_1(\Omega, \Sigma, \mathbb{P}, G)$  ( $:= L_1(\Omega, G)$ ) for all  $f \in F$ . Then define  $\tilde{S} : F \rightarrow L_1(\Omega, G)$  by setting

$$(\tilde{S}(f))(\omega) = S(f, \omega) \quad (\omega \in \Omega) \quad (24)$$

and put

$$\tilde{\mathcal{P}} = (F, L_1(\Omega, G), \tilde{S}, K, \Lambda_F). \quad (25)$$

The behavior of the stochastic problem  $\mathcal{P}$  with respect to semi-adaptive deterministic algorithms is related to that of  $\tilde{\mathcal{P}}$  in the deterministic setting as follows.

**Lemma 4.1.** *Let  $A$  be a semi-adaptive algorithm for  $\mathcal{P}$  and let  $N_1$  be given by (12). Then*

$$e(S, A, F \times \Omega, G) \geq \sup_{f, g \in F, N_1(f) = N_1(g)} \mathbb{E} \|\tilde{S}(f) - \tilde{S}(g)\|_{L_1(\Omega, G)}. \quad (26)$$

Moreover, for  $n \in \mathbb{N}_0$ ,

$$e_n^{\text{dsa}}(S, F \times \Omega, G) \geq \frac{1}{2} e_n^{\text{det}}(\tilde{S}, F, L_1(\Omega, G)). \quad (27)$$

*Proof.* We have

$$\begin{aligned} e(S, A, F \times \Omega, G) &= \sup_{f \in F} \mathbb{E} \|S(f, \omega) - A(f, \omega)\|_G \\ &= \frac{1}{2} \sup_{f, g \in F, N_1(f) = N_1(g)} \mathbb{E} (\|S(f, \omega) - A(f, \omega)\|_G + \|S(g, \omega) - A(g, \omega)\|_G) \\ &\geq \frac{1}{2} \sup_{f, g \in F, N_1(f) = N_1(g)} \mathbb{E} \|S(f, \omega) - S(g, \omega)\|_G \\ &= \frac{1}{2} \sup_{f, g \in F, N_1(f) = N_1(g)} \|\tilde{S}(f) - \tilde{S}(g)\|_{L_1(\Omega, G)}, \end{aligned} \quad (28)$$

which shows (26).

Now let  $n \in \mathbb{N}_0$  and assume that  $A$  satisfies, in addition,  $\text{card}(A, F) \leq n$ . Then the right-hand side of (26) is known to be connected with the deterministic  $n$ -th minimal error of  $\tilde{S}$  as follows, see also [17], Ch. 4.2. We define a mapping  $\tilde{\varphi} : K^\infty \rightarrow L_1(\Omega, G)$  by choosing for each  $a \in N_1(F)$  an  $f_a \in F$  with  $N_1(f_a) = a$  and setting  $\tilde{\varphi}(a) = \tilde{S}(f_a)$ . For  $a \in K^\infty \setminus N_1(F)$  we set  $\tilde{\varphi}(a) = 0$ . Consequently,

$$\begin{aligned} \sup_{f, g \in F, N_1(f) = N_1(g)} \|\tilde{S}(f) - \tilde{S}(g)\|_{L_1(\Omega, G)} &= \sup_{a \in N_1(F)} \sup_{f, g \in F, N_1(f) = N_1(g) = a} \|\tilde{S}(f) - \tilde{S}(g)\|_{L_1(\Omega, G)} \\ &\geq \sup_{a \in N_1(F)} \sup_{f \in F, N_1(f) = a} \|\tilde{S}(f) - \tilde{S}(f_a)\|_{L_1(\Omega, G)} \\ &= \sup_{a \in N_1(F)} \sup_{f \in F, N_1(f) = a} \|\tilde{S}(f) - \tilde{\varphi}(N_1(f))\|_{L_1(\Omega, G)} \\ &= \sup_{f \in F} \|\tilde{S}(f) - \tilde{\varphi}(N_1(f))\|_{L_1(\Omega, G)}. \end{aligned} \quad (29)$$

We claim that  $\tilde{\varphi} \circ N_1$  is the representation (8) of a suitable deterministic algorithm  $\tilde{A} = ((\tilde{L}_i)_{i=1}^\infty, (\tilde{\tau}_i)_{i=0}^\infty, (\tilde{\varphi}_i)_{i=1}^\infty)$  for  $\tilde{\mathcal{P}}$ . This is easily checked by taking any mapping  $\pi : \Lambda_F \cup \Lambda_\Omega \rightarrow \Lambda_F$  which is the identity on  $\Lambda_F$  and setting

$$\tilde{L}_{i+1} = \pi \circ L_i, \quad \tilde{\tau}_i = \sigma_i, \quad \tilde{\varphi}_i = \tilde{\varphi}|_{K^i} \quad (i \in \mathbb{N}_0).$$

It follows from the assumption on  $A$  and (11) that  $\text{card}(\tilde{A}, F) = \text{card}_F(A, F) \leq n$ , hence

$$\sup_{f \in F} \|S(f, \omega) - \tilde{\varphi}(N_1(f))\|_{L_1(\Omega, G)} \geq e_n^{\det}(\tilde{S}, F, L_1(\Omega, G)). \quad (30)$$

Combining (28) with (30) and taking the infimum over all  $N_1$  gives (27). □

Let  $(\gamma_j)_{j=1}^\infty$  be a sequence of independent standard Gaussian random variables. The case  $p = 2$  of the following lemma was shown in [7]. The proof for general  $p$  follows similar lines.

**Lemma 4.2.** *Let  $0 < p < \infty$ . Then there is a constant  $c(p) > 0$  such that for all  $m \in \mathbb{N}$*

$$\mathbb{P} \left\{ \omega \in \Omega : \min_{\mathcal{J} \subseteq \{1, \dots, m\}, |\mathcal{J}| \geq m/2} \left( \sum_{j \in \mathcal{J}} |\gamma_j(\omega)|^p \right)^{1/p} \geq c(p)m^{1/p} \right\} \geq 7/8.$$

*Proof.* Let  $B_p^k$  denote the unit ball of  $\mathbb{R}^k$ , endowed with the  $\ell_p$  (quasi-)norm  $\|(x_1, \dots, x_k)\|_{\ell_p} = \left( \sum_{i=1}^k |x_i|^p \right)^{1/p}$ . There is a constant  $c_0(p) > 0$  such that for all  $k \in \mathbb{N}$

$$\text{Vol}(B_p^k) \leq c_0(p)^k k^{-k/p}, \quad (31)$$

see, e.g., [14], relation 1.18 on p. 11. Define  $c(p) = 2^{-5-1/p} c_0(p)^{-1}$ . Let  $m \in \mathbb{N}$  and set  $k = \lceil m/2 \rceil$ . Then

$$\begin{aligned} & \mathbb{P} \left\{ \min_{\mathcal{J} \subseteq \{1, \dots, m\}, |\mathcal{J}| \geq m/2} \sum_{j \in \mathcal{J}} |\gamma_j(\omega)|^p \geq c(p)^p m \right\} \\ &= \mathbb{P} \left\{ \min_{\mathcal{J} \subseteq \{1, \dots, m\}, |\mathcal{J}| = k} \sum_{j \in \mathcal{J}} |\gamma_j(\omega)|^p \geq c(p)^p m \right\} \\ &\geq 1 - \sum_{\mathcal{J} \subseteq \{1, \dots, m\}, |\mathcal{J}| = k} \mathbb{P} \left\{ \sum_{j \in \mathcal{J}} |\gamma_j(\omega)|^p < c(p)^p m \right\} \\ &\geq 1 - 2^{2k} \mathbb{P} \left\{ \sum_{j=1}^k |\gamma_j(\omega)|^p < 2c(p)^p k \right\}. \end{aligned} \quad (32)$$

Furthermore, using (31),

$$\begin{aligned} \mathbb{P} \left\{ \sum_{j=1}^k |\gamma_j(\omega)|^p < 2c(p)^p k \right\} &= (2\pi)^{-k/2} \int_{\|x\|_{\ell_p} \leq c(p)(2k)^{1/p}} e^{-|x|^2/2} dx \\ &\leq \text{Vol}(c(p)(2k)^{1/p} B_p^k) = c(p)^k (2k)^{k/p} \text{Vol}(B_p^k) \\ &\leq 2^{k/p} c(p)^k c_0(p)^k \leq 2^{-5k}. \end{aligned} \quad (33)$$



From (32) and (33) we conclude

$$\mathbb{P} \left\{ \min_{\mathcal{J} \subseteq \{1, \dots, m\}, |\mathcal{J}| \geq m/2} \sum_{j \in \mathcal{J}} |\gamma_j(\omega)|^p \geq c(p)^p m \right\} \geq 1 - 2^{-3k} \geq 7/8.$$

□

Let  $\Lambda' \subseteq \Lambda$  be any nonempty subset such that  $\Lambda \setminus \Lambda' \subseteq \Lambda_\Omega$ . For  $\omega \in \Omega$  let the mapping  $S_\omega : F \rightarrow G$  be given by

$$S_\omega(f) = S(f, \omega) \quad (f \in F) \quad (34)$$

and define the restricted problem  $\mathcal{P}_\omega = (F, G, S_\omega, K, \Lambda_\omega)$ , where

$$\Lambda_\omega = \{\lambda(\cdot, \omega) : \lambda \in \Lambda'\}.$$

The following lower bound for the randomized  $n$ -th minimal errors was shown in [7], Lemma 4. Here  $\int^*$  denotes the upper integral.

**Lemma 4.3.** *Let  $\nu$  be a probability measure on  $F$  supported by a finite set. Then for all  $n \in \mathbb{N}_0$ ,*

$$e_n^{\text{ran}}(S, F \times \Omega, G) \geq \frac{1}{3} \inf_{D \in \Sigma, \mathbb{P}(D) \geq 1/4} \int_D^* e_{2n}^{\text{avg}}(S_\omega, \nu, G) d\mathbb{P}(\omega). \quad (35)$$

Now we will use the results from Section 3. Let  $D, Z, F_1, F$  be as in Section 3 and assume that for each  $\omega \in \Omega$ , the mapping  $S_\omega : F \rightarrow G$  given by (34) is the restriction to  $F$  of a linear operator from  $F_1$  to  $G$ . Moreover, suppose there is a set  $\Lambda'$  such that  $\emptyset \neq \Lambda' \subseteq \Lambda$ ,  $\Lambda \setminus \Lambda' \subseteq \Lambda_\Omega$ , and for each  $\lambda \in \Lambda'$  and each  $\omega \in \Omega$  there is an  $x \in D$  such that

$$\lambda(f, \omega) = \delta_x(f) \quad (f \in F).$$

Combining Lemma 4.3 with Proposition 3.1, we obtain

**Proposition 4.4.** *Let  $\bar{n}, n \in \mathbb{N}$ ,  $\bar{n} > 8n$ , and assume that there are  $\{w_i\}_{i=1}^{\bar{n}} \subset F$  so that (16) is satisfied.*

1. *Assume that  $\sum_{i=1}^{\bar{n}} \beta_i w_i \in F$  for all  $\beta_i \in \{-1, 1\}$ ,  $i = 1, \dots, \bar{n}$ . Let  $(\varepsilon_i)_{i=1}^{\bar{n}}$  be independent, centered Bernoulli random variables on another probability space  $(\Omega_1, \Sigma_1, \mathbb{P}_1)$ . Then*

$$e_n^{\text{ran}}(S, F \times \Omega, G) \geq \frac{1}{6} \inf_{D \in \Sigma, \mathbb{P}(D) \geq 1/4} \int_D \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n} - 4n} \mathbb{E}_1 \left\| \sum_{k \in K} \varepsilon_k S(w_k, \omega) \right\|_G d\mathbb{P}(\omega). \quad (36)$$

2. *Assume that  $-w_i \in F$  for  $i = 1, \dots, \bar{n}$ . Then*

$$e_n^{\text{ran}}(S, F \times \Omega, G) \geq \frac{1}{3\bar{n}} \inf_{D \in \Sigma, \mathbb{P}(D) \geq 1/4} \int_D \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n} - 4n} \sum_{k \in K} \|S(w_k, \omega)\|_G. \quad (37)$$

**Corollary 4.5.** *Let  $G = \mathbb{R}$ . Then there is a constant  $c > 0$  such that the following holds. Under the conditions of Proposition 4.4, assume that the random variables  $S(w_i, \omega)$  ( $i = 1, \dots, \bar{n}$ ) are independent and Gaussian of mean zero. Then in case 1 of Proposition 4.4*

$$e_n^{\text{ran}}(S, F \times \Omega, \mathbb{R}) \geq c\bar{n}^{1/2} \min_{1 \leq i \leq \bar{n}} (\mathbb{E} |S(w_i, \omega)|^2)^{1/2}, \quad (38)$$

while in case 2

$$e_n^{\text{ran}}(S, F \times \Omega, \mathbb{R}) \geq c \min_{1 \leq i \leq \bar{n}} (\mathbb{E} |S(w_i, \omega)|^2)^{1/2}. \quad (39)$$

*Proof.* Set  $\sigma_i = (\mathbb{E} |S(w_i, \omega)|^2)^{1/2}$ ,  $S(w_i, \omega) = \sigma_i \gamma_i(\omega)$ . Let  $c(p)$  for  $p = 1, 2$  be the constants from Lemma 4.2. First consider case 1 of Proposition 4.4. Then by Khintchine's inequality, see [11], there is a constant  $c_0 > 0$  such that

$$\mathbb{E}_1 \left| \sum_{k \in K} \varepsilon_k S(w_k, \omega) \right| \geq c_0 \left( \sum_{k \in K} S(w_k, \omega)^2 \right)^{1/2}.$$

Using this and Lemma 4.2, we obtain

$$\begin{aligned} & \mathbb{P} \left\{ \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n} - 4n} \mathbb{E}_1 \left| \sum_{k \in K} \varepsilon_k S(w_k, \omega) \right| \geq c_0 c(2) \bar{n}^{1/2} \min_{1 \leq i \leq \bar{n}} \sigma_i \right\} \\ & \geq \mathbb{P} \left\{ \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n} - 4n} \left( \sum_{k \in K} S(w_k, \omega)^2 \right)^{1/2} \geq c(2) \bar{n}^{1/2} \min_{1 \leq i \leq \bar{n}} \sigma_i \right\} \\ & = \mathbb{P} \left\{ \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n}/2} \left( \sum_{k \in K} \sigma_k^2 \gamma_k(\omega)^2 \right)^{1/2} \geq c(2) \bar{n}^{1/2} \min_{1 \leq i \leq \bar{n}} \sigma_i \right\} \\ & \geq \mathbb{P} \left\{ \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n}/2} \left( \sum_{k \in K} \gamma_k(\omega)^2 \right)^{1/2} \geq c(2) \bar{n}^{1/2} \right\} \geq \frac{7}{8}, \end{aligned}$$

which together with (36) implies (38). In case 2 we argue similarly:

$$\begin{aligned} & \mathbb{P} \left\{ \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n} - 4n} \sum_{k \in K} |S(w_k, \omega)| \geq c(1) \bar{n} \min_{1 \leq i \leq \bar{n}} \sigma_i \right\} \\ & = \mathbb{P} \left\{ \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n}/2} \sum_{k \in K} \sigma_k |\gamma_k(\omega)| \geq c(1) \bar{n} \min_{1 \leq i \leq \bar{n}} \sigma_i \right\} \\ & \geq \mathbb{P} \left\{ \min_{K \subseteq \{1, \dots, \bar{n}\}, |K| \geq \bar{n}/2} \sum_{k \in K} |\gamma_k(\omega)| \geq c(1) \bar{n} \right\} \geq \frac{7}{8}. \end{aligned}$$

With (37) this implies (39). □

## 5 Stochastic integration

Now we consider stochastic integration. We restrict our analysis to deterministic integrands. Let  $Q \subset \mathbb{R}^d$  be a bounded Lipschitz domain, that is, a set which is the closure of a bounded open set with locally Lipschitz boundary. Let  $Q$  be equipped with the Lebesgue measure. Let  $1 \leq p \leq \infty$  and let  $L_p(Q)$  denote the space of equivalence classes of real-valued, Borel measurable,  $p$ -integrable functions, equipped with the norm

$$\|f\|_{L_p(Q)} = \left( \int_Q |f(x)|^p dx \right)^{1/p}$$

for  $p < \infty$ , and

$$\|f\|_{L_\infty(Q)} = \text{ess sup}_{x \in Q} |f(x)|.$$

In the sequel it is convenient to distinguish between the space of equivalence classes of functions  $L_p(Q)$  and the linear space of all functions  $\mathcal{L}_p(Q)$ , thus  $f \in \mathcal{L}_p(Q)$  iff  $[f] \in L_p(Q)$ , where  $[f]$  is the equivalence class of  $f$  with respect to equality up to a subset of  $Q$  of Lebesgue measure zero. The latter is a linear space and  $\|[f]\|_{L_p(Q)}$  is a semi-norm on it (we use though the same symbol  $\|f\|_{L_p(Q)}$  for it).

Let  $\hat{W}_Q$  be an  $L_2(Q)$ -isonormal process on some probability space  $(\Omega, \Sigma, \mathbb{P})$ , that is, a linear isometric operator from  $L_2(Q)$  to  $L_2(\Omega)$  such that  $\hat{W}_Q(f)$  is standard Gaussian for all  $f \in L_2(Q)$  with  $\|f\|_{L_2(Q)} = 1$  (see, e.g., [13]). The stochastic integral of  $f \in L_2(Q)$  with respect to  $\hat{W}_Q$  is defined as

$$\int_Q f(x) d\hat{W}_Q := \hat{W}_Q(f). \quad (40)$$

For our purposes we need a suitable pathwise defined version

$$W_Q : \mathcal{L}_2(Q) \times \Omega \rightarrow \mathbb{R} \quad (41)$$

such that  $W_Q(f, \omega)$  is linear in  $f$  for each  $\omega \in \Omega$ , and for each  $f \in \mathcal{L}_2(Q)$

$$W_Q(f, \cdot) = \hat{W}_Q(f), \quad (42)$$

with equality meant in  $L_2(\Omega)$ . Let  $(f_i)_{i \in I}$ ,  $I$  a suitable index set, be a Hamel basis of  $L_2(Q)$ . For each  $i \in I$  let  $g_i = g_i(\omega)$  be a representative of the equivalence class  $\hat{W}_Q(f_i) \in L_2(\Omega)$ . Then we set  $\tilde{W}_Q(f_i, \omega) = g_i(\omega)$  for  $i \in I$  and  $\omega \in \Omega$  and extend the so-defined mapping by linearity to all of  $L_2(Q)$ . Finally, we define  $W_Q(f, \omega) = \tilde{W}_Q([f], \omega)$  for  $f \in \mathcal{L}_2(Q)$ . It follows from the linearity of  $\hat{W}_Q$  that  $W_Q$  is as required.

Now let  $F$  be any nonempty subset of  $\mathcal{L}_2(Q)$ ,  $G = \mathbb{R}$ ,  $K = \mathbb{R} \cup K_2$ , where  $K_2$  is any non-empty set, and

$$\Lambda = \Lambda_1 \cup \Lambda_2, \quad (43)$$

where  $\Lambda_1 = \{\delta_t^1 : t \in Q\}$ , with  $\delta_t^1(f, \omega) = f(t)$ , and  $\Lambda_2$  is a non-empty set of mappings from  $\Omega$  to  $K_2$ . Thus, formulated in the terminology above, we consider the problem

$$\mathcal{P}_Q = (F, (\Omega, \Sigma, \mathbb{P}), \mathbb{R}, W_Q, K, \Lambda). \quad (44)$$

Note that with  $D = Q$ ,  $Z = \mathbb{R}$ ,  $\Lambda' = \Lambda_1$  the assumptions formulated before Proposition 4.4 are satisfied.

Let  $Q_0 \subseteq Q$ ,

$$Q_0 = \prod_{i=1}^d [a_i, a_i + b]. \quad (45)$$

be a cube of side-length  $b > 0$ , let  $m \in \mathbb{N}$  and let

$$Q_0 = \bigcup_{j=1}^{m^d} Q_{0,j}$$

be the partition of  $Q_0$  into  $m^d$  congruent cubes of disjoint interior. Let  $x_j$  denote the point in  $Q_{0,j}$  with minimal coordinates. Furthermore, let  $\psi$  be a  $C^\infty$  function on  $\mathbb{R}^d$  with support in  $[0, 1]^d$  satisfying  $\|\psi\|_{L_2(\mathbb{R}^d)} = 1$ . Define for  $x \in Q$

$$\psi_j(x) = \psi(b^{-1}m(x - x_j)) \quad (j = 1, \dots, m^d). \quad (46)$$

We have

$$(\mathbb{E} |W_Q(\psi_j, \omega)|^2)^{1/2} = \|\psi_j\|_{L_2(Q)} = b^{d/2} m^{-d/2}. \quad (47)$$

Denote

$$\Psi_m^1 = \left\{ \sum_{j=1}^{m^d} \alpha_j \psi_j : \alpha_j \in \{-1, 1\} \right\}, \quad \Psi_m^2 = \{\beta \psi_j : \beta \in \{-1, 1\}, 1 \leq j \leq m^d\}.$$

Finally, let  $J_F : F \rightarrow L_2(Q)$  be the embedding map.

**Proposition 5.1.** *There are constants  $c_1, c_2, c_3 > 0$  such that the following hold: for all  $m, n \in \mathbb{N}$  with  $m^d > 8n$*

$$e_n^{\text{ran}}(W_Q, F \times \Omega, \mathbb{R}) \geq c_1 \sup\{\kappa \geq 0 : \kappa \Psi_m^1 \subseteq F\} \quad (48)$$

$$e_n^{\text{ran}}(W_Q, F \times \Omega, \mathbb{R}) \geq c_2 m^{-d/2} \sup\{\kappa \geq 0 : \kappa \Psi_m^2 \subseteq F\} \quad (49)$$

and for all  $n \in \mathbb{N}$

$$e_n^{\text{dsa}}(W_Q, F \times \Omega, \mathbb{R}) \geq c_3 e_n^{\text{det}}(J_F, F, L_2(Q)). \quad (50)$$

Note that Proposition 5.1 holds for arbitrary  $\Lambda_2$ , see (43), including the (strongest) case of full information about  $\omega$ , that is,  $K_2 = \Omega$  and  $\Lambda_2 = \{Id_\Omega\}$ . Relation (50) is due to Przybyłowicz [15].

*Proof.* Let  $m, n \in \mathbb{N}$ ,  $m^d > 8n$ . Let  $\kappa \geq 0$ , to be fixed later on and put  $w_j = \kappa \psi_j$  for  $j = 1, \dots, m^d$ . By (47) and the linearity of  $W_Q$  in the first argument,

$$(\mathbb{E} |W_Q(w_j, \omega)|^2)^{1/2} = \kappa (\mathbb{E} |W_Q(\psi_j, \omega)|^2)^{1/2} = \kappa b^{d/2} m^{-d/2}. \quad (51)$$

Moreover, the random variables  $W_Q(w_j, \omega)$  ( $j = 1, \dots, m^d$ ) are independent and Gaussian of mean zero. To show (48), let  $\kappa$  be such that  $\kappa \Psi_m^1 \subseteq F$ . From (38) of Corollary 4.5 with  $\bar{n} = m^d$  we conclude, using also (51),

$$e_n^{\text{ran}}(W_Q, F \times \Omega, \mathbb{R}) \geq c m^{d/2} \min_{1 \leq i \leq \bar{n}} (\mathbb{E} |W_Q(w_i, \omega)|^2)^{1/2} \geq c \kappa, \quad (52)$$

showing (48). Now let  $\kappa$  be such that  $\kappa \Psi_m^2 \subseteq F$ . Using (39) and (51), we get

$$e_n^{\text{ran}}(W_Q, F \times \Omega, \mathbb{R}) \geq c \min_{1 \leq i \leq \bar{n}} (\mathbb{E} |W_Q(w_i, \omega)|^2)^{1/2} \geq c \kappa m^{-d/2}, \quad (53)$$

which gives (49).

Finally, to show (50), let  $A$  be a deterministic semi-adaptive algorithm for  $\mathcal{P}_Q$  with  $\text{card}(A, F) \leq n$  and let  $N_1$  be given by (12). It follows from (26) of Lemma 4.1 that

$$\begin{aligned} e(W_Q, A, F \times \Omega, \mathbb{R}) &\geq \sup_{f, g \in F, N_1(f) = N_1(g)} \|W_Q(f, \cdot) - W_Q(g, \cdot)\|_{L_1(\Omega)} \\ &\geq c \sup_{f, g \in F, N_1(f) = N_1(g)} \|f - g\|_{L_2(Q)}. \end{aligned} \quad (54)$$

Arguing as in the proof of (27) of Lemma 4.1 yields (50). □

Next we give a general upper bound. Here we specify  $\Lambda_2$  from (43) to be the class  $\Lambda_{2,0}$  of linear functionals of the stochastic process, more precisely

$$\Lambda_{2,0} = \{\delta_h^2 : h \in L_2(Q)\}, \quad (55)$$

where

$$\delta_h^2(f, \omega) = W_Q(h, \omega) \quad (f \in F, \omega \in \Omega). \quad (56)$$

Thus,  $K = K_2 = \mathbb{R}$ . Fix a finite dimensional subspace  $H \subset L_2(Q)$ . Let  $n \in \mathbb{N}$ ,  $x_i \in Q$ ,  $g_i \in H$  ( $i = 1, \dots, n$ ) and define  $T : F \rightarrow L_2(Q)$  by

$$Tf = \sum_{i=1}^n f(x_i)g_i \quad (f \in F).$$

Moreover, let  $(\Omega_1, \Sigma_1, \mathbb{P}_1)$  be another probability space, let  $\xi_i$  and  $\zeta_i$  be random variables on  $(\Omega_1, \Sigma_1, \mathbb{P}_1)$  with values in  $Q$  and  $H$ , respectively. For  $\omega_1 \in \Omega_1$  define

$$T_{\omega_1}f = \sum_{i=1}^n f(\xi_i(\omega_1))\zeta_i(\omega_1) \quad (f \in F).$$

**Lemma 5.2.**

$$e_{2n}^{\text{dsa}}(W_Q, F \times \Omega, \mathbb{R}) \leq \sup_{f \in F} \|f - Tf\|_{L_2(Q)} \quad (57)$$

$$e_{2n}^{\text{ran}}(W_Q, F \times \Omega, \mathbb{R}) \leq \sup_{f \in F} (\mathbb{E}_{\mathbb{P}_1} \|f - T_{\omega_1}f\|_{L_2(Q)}^2)^{1/2}. \quad (58)$$

Suppose that  $F$  is relatively compact in  $L_2(Q)$  and there is an  $m \in \mathbb{N}$  and Borel measurable mappings  $\Phi_i : \mathbb{R}^m \rightarrow Q$ ,  $\Theta_i : \mathbb{R}^m \rightarrow H$  ( $i = 1, \dots, n$ ), such that if  $(\gamma_j)_{j=1}^m$  is a sequence of independent standard Gaussian random variables, then  $((\Phi_i(\gamma_1, \dots, \gamma_m))_{i=1}^n, (\Theta_i(\gamma_1, \dots, \gamma_m))_{i=1}^n)$  and  $((\xi_i(\omega_1))_{i=1}^n, (\zeta_i(\omega_1))_{i=1}^n)$  have the same distribution. Then

$$e_{m+2n}^{\text{det}}(W_Q, F \times \Omega, \mathbb{R}) \leq \sup_{f \in F} (\mathbb{E}_{\mathbb{P}_1} \|f - T_{\omega_1}f\|_{L_2(Q)}^2)^{1/2}. \quad (59)$$

*Proof.* We define the algorithm  $A_0$  for  $f \in F$  and  $\omega \in \Omega$  by

$$A_0(f, \omega) = W_Q(Tf, \omega) = \sum_{i=1}^n f(x_i)W_Q(g_i, \omega).$$

Clearly,  $A_0 \in \mathcal{A}_{2n}^{\text{dsa}}(\mathcal{P}_Q)$  and

$$\begin{aligned} e(W_Q, A_0, F \times \Omega, \mathbb{R}) &= \sup_{f \in F} \mathbb{E} |W_Q(f, \omega) - A_0(f, \omega)| \\ &\leq \sup_{f \in F} (\mathbb{E} |W_Q(f, \omega) - W_Q(Tf, \omega)|^2)^{1/2} = \sup_{f \in F} \|f - Tf\|_{L_2(Q)}, \end{aligned}$$

which gives (57). Now let  $(h_l)_{l=1}^M$  be an orthonormal basis of  $H$ . For  $f \in F$ ,  $\omega \in \Omega$ , and  $\omega_1 \in \Omega_1$  we set

$$A_{\omega_1}(f, \omega) = W_Q(T_{\omega_1}f, \omega) = \sum_{i=1}^n f(\xi_i(\omega_1))W_Q(\zeta_i(\omega_1), \omega) \quad (60)$$

$$= \sum_{i=1}^n \sum_{l=1}^M f(\xi_i(\omega_1))(\zeta_i(\omega_1), h_l)W_Q(h_l, \omega). \quad (61)$$

Then  $A = ((\Omega_1, \Sigma_1, \mathbb{P}_1), (A_{\omega_1})_{\omega_1 \in \Omega_1})$  belongs to  $\mathcal{A}_{2n}^{\text{ran}}(\mathcal{P}_Q)$ . Indeed, the statement about the cardinality is a consequence of representation (60), while measurability follows from (61). Similarly to the deterministic case we conclude

$$\begin{aligned} e(W_Q, A, F \times \Omega, \mathbb{R})^2 &\leq \sup_{f \in F} \mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}} |W_Q(f, \omega) - A_{\omega_1}(f, \omega)|^2 \\ &= \sup_{f \in F} \mathbb{E}_{\mathbb{P}_1} \mathbb{E}_{\mathbb{P}} |W_Q(f, \omega) - W_Q(T_{\omega_1} f, \omega)|^2 \\ &= \sup_{f \in F} \mathbb{E}_{\mathbb{P}_1} \|f - T_{\omega_1} f\|_{L_2(Q)}^2. \end{aligned}$$

This implies (58).

To show (59), let  $\delta > 0$  and let  $F_0 \subseteq F$  be a finite  $\delta$ -net of  $F$ , i.e.,

$$\sup_{f \in F} \min_{g \in F_0} \|f - g\|_{L_2(Q)} \leq \delta. \quad (62)$$

We represent the randomized algorithm  $A$  above as a deterministic algorithm  $\tilde{A}$  which exploits the randomness of the underlying probability space  $(\Omega, \Sigma, \mathbb{P})$ . Let  $(g_j)_{j=1}^m$  be an orthonormal system in  $L_2(Q)$  orthogonal to  $\text{span}(H, F_0)$ . We define  $\tilde{A} = ((\tilde{L}_i)_{i=1}^\infty, (\tilde{\tau}_i)_{i=0}^\infty, (\tilde{\varphi}_i)_{i=1}^\infty)$  as follows. For  $i \in \mathbb{N}$  and  $(z_1, \dots, z_{i-1}) \in \mathbb{R}^i$  put

$$\tilde{L}_i(z_1, \dots, z_{i-1}) = \begin{cases} \delta_{g_i}^2 & \text{if } 1 \leq i \leq m \\ \delta_{\Phi_{i-m}(z_1, \dots, z_m)}^1 & \text{if } m < i \leq m+n \\ \delta_{\Theta_{i-m-n}(z_1, \dots, z_m)}^2 & \text{if } m+n < i \leq m+2n \\ \delta_0^2 & \text{if } m+2n < i. \end{cases} \quad (63)$$

Furthermore, we set

$$\tilde{\tau}_1 = \dots = \tilde{\tau}_{m+2n-1} = 0, \quad \tilde{\tau}_{m+2n} = 1, \quad \tilde{\tau}_i = 0 \quad (i > m+2n) \quad (64)$$

$$\tilde{\varphi}_{m+2n}(z_1, \dots, z_{m+2n}) = \sum_{j=1}^n z_{m+j} z_{m+n+j}, \quad \tilde{\varphi}_i \equiv 0 \quad (i \neq m+2n). \quad (65)$$

Now we fix  $f \in F$ . Then for each  $\omega \in \Omega$

$$a_i := (\tilde{L}_i(a_1, \dots, a_{i-1}))(f, \omega) = \begin{cases} W_Q(g_i, \omega) & \text{if } 1 \leq i \leq m \\ f(\Phi_{i-m}(a_1, \dots, a_m)) & \text{if } m < i \leq m+n \\ W_Q(\Theta_{i-m-n}(a_1, \dots, a_m), \omega) & \text{if } m+n < i \leq m+2n. \end{cases}$$

Note that  $a_1, \dots, a_m$  are independent standard Gaussian random variables. Moreover,

$$\begin{aligned} \tilde{A}(f, \omega) &= \sum_{j=1}^n a_{m+j} a_{m+n+j} = \sum_{j=1}^n f(\Phi_j(a_1, \dots, a_m)) W_Q(\Theta_j(a_1, \dots, a_m), \omega) \\ &= \sum_{j=1}^n \sum_{l=1}^M f(\Phi_j(a_1, \dots, a_m)) (\Theta_j(a_1, \dots, a_m), h_l) W_Q(h_l, \omega). \end{aligned} \quad (66)$$

It follows that  $\tilde{A} \in \mathcal{A}_{m+2n}^{\text{det}}(\mathcal{P}_Q)$ . Let  $f_0 \in F_0$  be such that  $\|f - f_0\|_{L_2(Q)} \leq \delta$ . Then

$$\mathbb{E}_{\mathbb{P}} |W_Q(f, \omega) - \tilde{A}(f, \omega)| \leq \mathbb{E}_{\mathbb{P}} |W_Q(f_0, \omega) - \tilde{A}(f, \omega)| + \delta. \quad (67)$$

The following relations hold. To justify the step from (68) to (69) below, we note that the  $(g_j)_{j=1}^m$  are orthogonal to  $f_0$  and  $(h_l)_{l=1}^M$ , hence the random variables  $(a_j)_{j=1}^m = (W_Q(g_j, \omega))_{j=1}^m$  are independent of  $W_Q(f_0, \omega)$  and  $(W_Q(h_l, \omega))_{l=1}^M$ . The step from (70) to (71) uses the distribution assumption of the lemma.

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}}(W_Q(f_0, \omega) - \tilde{A}(f, \omega))^2 \\ &= \mathbb{E}_{\mathbb{P}} \left( W_Q(f_0, \omega) - \sum_{j=1}^n \sum_{l=1}^M f(\Phi_j(a_1, \dots, a_m))(\Theta_j(a_1, \dots, a_m), h_l) W_Q(h_l, \omega) \right)^2 \end{aligned} \quad (68)$$

$$= \mathbb{E}_{\mathbb{P}} \left\| f_0 - \sum_{j=1}^n \sum_{l=1}^M f(\Phi_j(a_1, \dots, a_m))(\Theta_j(a_1, \dots, a_m), h_l) h_l \right\|_{L_2(Q)}^2 \quad (69)$$

$$= \mathbb{E}_{\mathbb{P}} \left\| f_0 - \sum_{j=1}^n f(\Phi_j(a_1, \dots, a_m)) \Theta_j(a_1, \dots, a_m) \right\|_{L_2(Q)}^2 \quad (70)$$

$$= \mathbb{E}_{\mathbb{P}_1} \left\| f_0 - \sum_{j=1}^n f(\xi_j(\omega_1)) \zeta_j(\omega_1) \right\|_{L_2(Q)}^2 \quad (71)$$

$$= \mathbb{E}_{\mathbb{P}_1} \|f_0 - T_{\omega_1} f\|_{L_2(Q)}^2 \leq \mathbb{E}_{\mathbb{P}_1} (\|f - T_{\omega_1} f\|_{L_2(Q)} + \delta)^2. \quad (72)$$

Since  $\delta > 0$  was arbitrary, (67) and (72) imply (59). □

## 6 Stochastic integration of functions from Sobolev spaces

Let  $1 \leq p \leq \infty$ . For  $r \in \mathbb{N}$ , the Sobolev space  $W_p^r(Q)$  consists of all equivalence classes of functions  $f \in L_p(Q)$  such that for all  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$  with  $|\alpha| := \sum_{j=1}^d \alpha_j \leq r$ , the generalized partial derivative  $D^\alpha f$  belongs to  $L_p(Q)$ . The norm on  $W_p^r(Q)$  is defined as

$$\|f\|_{W_p^r(Q)} = \left( \sum_{|\alpha| \leq r} \|D^\alpha f\|_{L_p(Q)}^p \right)^{1/p} \quad (73)$$

if  $p < \infty$ , and

$$\|f\|_{W_\infty^r(Q)} = \max_{|\alpha| \leq r} \|D^\alpha f\|_{L_\infty(Q)}. \quad (74)$$

For  $r = 0$  we set  $W_p^0(Q) := L_p(Q)$ . Let  $C(Q)$  denote the space of continuous functions on  $Q$ , endowed with the supremum norm. If  $r/d > \max(1/p - 1/2, 0)$ , then  $W_p^r(Q)$  is compactly embedded into  $L_2(Q)$ , and we denote the respective embedding map by  $J_p^r$ . Furthermore,  $W_p^r(Q)$  is continuously embedded into  $C(Q)$  if and only if

$$\left. \begin{array}{l} p = 1 \quad \text{and} \quad r/d \geq 1 \\ \text{or} \\ 1 < p \leq \infty \quad \text{and} \quad r/d > 1/p, \end{array} \right\} \quad (75)$$

see [1], Ch. 5.

We remain in the same framework as described in Section 5, see (43) and (44). If (75) holds and the deterministic semi-adaptive setting is concerned, we consider  $W_p^r(Q)$  as identified with a subset of  $C(Q)$ , hence, function values at points of  $Q$  are well-defined. Here we set  $F = B_{W_p^r(Q)}$ . In the other cases it is convenient to consider also the respective Sobolev spaces of functions, which we denote by  $\mathcal{W}_p^r(Q)$ , thus  $f \in \mathcal{W}_p^r(Q)$  iff  $[f] \in W_p^r(Q)$ . If we are in the deterministic adaptive or in the randomized setting, we put  $F = B_{\mathcal{W}_p^r(Q)}$ . Finally, if the deterministic semi-adaptive setting is considered and (75) does not hold, the lower bound statement  $e_n^{\text{dsa}}(W_Q, B_{\mathcal{W}_p^r(Q)} \times \Omega, \mathbb{R}) \geq 1$  would be trivial, because function values at finitely many deterministic points could be arbitrarily perturbed without changing the class of the function and thus without changing the solution. Here we consider, less trivially, the dense subset  $B_{\mathcal{W}_p^r(Q)} \cap C(Q)$  of  $B_{\mathcal{W}_p^r(Q)}$ . Relation (78) shows that even then no nontrivial rate can hold for deterministic semi-adaptive algorithms, that is, for algorithms that use deterministic sampling of the integrand.

The following result gives the  $n$ -th minimal errors in all three settings and thus the respective complexities.

**Theorem 6.1.** *Let  $d, r \in \mathbb{N}$ ,  $1 \leq p, q \leq \infty$ ,  $r/d > \max(1/p - 1/2, 0)$ . Then*

$$e_n^{\text{det}}(W_Q, B_{\mathcal{W}_p^r(Q)} \times \Omega, \mathbb{R}) \asymp e_n^{\text{ran}}(W_Q, B_{\mathcal{W}_p^r(Q)} \times \Omega, \mathbb{R}) \asymp n^{-\frac{r}{d} + \max(\frac{1}{p} - \frac{1}{2}, 0)} \quad (76)$$

$$e_n^{\text{dsa}}(W_Q, B_{\mathcal{W}_p^r(Q)} \times \Omega, \mathbb{R}) \asymp n^{-\frac{r}{d} + \max(\frac{1}{p} - \frac{1}{2}, 0)} \quad \text{if } \frac{r}{d} > \frac{1}{p} \vee \left( \frac{r}{d} = \frac{1}{p} \wedge p = 1 \right) \quad (77)$$

$$e_n^{\text{dsa}}(W_Q, (B_{\mathcal{W}_p^r(Q)} \cap C(Q)) \times \Omega, \mathbb{R}) \asymp 1 \quad \text{if } \frac{r}{d} < \frac{1}{p} \vee \left( \frac{r}{d} = \frac{1}{p} \wedge p > 1 \right). \quad (78)$$

To prove the upper bounds we will use approximation results from [6]. We recall some details of the construction, for full background we refer to [6]. For a set  $B$  let  $\mathcal{F}(B)$  denote the set of all real-valued functions on  $B$ . We start with randomly shifted interpolation on  $[0, 1]^d$ . Fix  $\varrho \in \mathbb{N}_0$ ,  $0 < \delta < 1$ , and let

$$Pf = \sum_{j=1}^{\kappa} f(z_j) \pi_j \quad (f \in \mathcal{F}([0, 1]^d))$$

be for  $d = 1$  the Lagrange interpolation operator of degree  $\varrho$  and for  $d > 1$  its tensor product, where  $(z_j)_{j=1}^{\kappa}$  is the uniform grid on  $[0, 1 - \delta]^d$  and  $(\pi_j)_{j=1}^{\kappa}$  are the respective Lagrange polynomials of degree  $d\varrho$  on  $\mathbb{R}^d$ . Let  $\theta$  be a uniformly distributed on  $[0, 1]^d$  random variable, defined on a probability space  $(\Omega_1, \Sigma_1, \mathbb{P}_1)$ . For a function  $f \in \mathcal{F}([0, 1]^d)$  put

$$(P_{\omega_1} f)(x) = \sum_{j=1}^{\kappa} f(z_j + \delta\theta(\omega_1)) \pi_j(x - \delta\theta(\omega_1)) \quad (x \in \mathbb{R}^d). \quad (79)$$

Now we consider the Lipschitz domain  $Q$  and apply the operator  $P_{\omega_1}$  locally. We let  $x_0 \in \mathbb{R}^d$  and  $b > 0$  be such that

$$Q \subset \tilde{Q} := x_0 + [0, b]^d. \quad (80)$$

For  $l \in \mathbb{N}_0$  let

$$\tilde{Q} = \bigcup_{i=1}^{2^{dl}} Q_{li}$$



be the partition of  $\tilde{Q}$  into  $2^{dl}$  cubes of sidelength  $b2^{-l}$  and of disjoint interior. Let  $x_{li}$  denote the point in  $Q_{li}$  with minimal coordinates and let

$$\mathcal{I}_l = \{i : 1 \leq i \leq 2^{dl}, Q_{li} \subseteq Q\}. \quad (81)$$

Finally, we fix a  $\sigma \in \mathbb{N}_0$  and set for  $f \in \mathcal{F}(Q)$

$$(P_{l,\omega_1}f)(x) = \sum_{i \in \mathcal{I}_l} \sum_{j=1}^{\kappa} f(x_{li} + b2^{-l}(z_j + \delta\theta(\omega_1))) \eta_{li}(x) \pi_j(b^{-1}2^l(x - x_{li}) - \delta\theta(\omega_1)) \quad (x \in Q). \quad (82)$$

Here  $\eta_{li}(x)$  are  $C^\sigma$  functions on  $\mathbb{R}^d$  of bounded support, forming a partition of unity on  $Q$ , see [6] for details. Moreover, let  $P_{l,0}$  be the (deterministic) operator which results from (82) when  $\theta(\omega_1)$  is replaced by 0. The following was shown in [6], Prop. 3.3 and 4.1.

**Proposition 6.2.** *Let  $r \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $r/d > \max(1/p - 1/2, 0)$ ,  $\varrho \geq r - 1$ ,  $\sigma \geq 0$ . Then there are constants  $c_1, c_2, c_3 > 0$ ,  $l_0 \in \mathbb{N}_0$  such that for all  $l \geq l_0$*

$$\sup_{f \in B_{W_p^r(Q)}} (\mathbb{E}_{\mathbb{P}_1} \|f - P_{l,\omega_1}f\|_{L_2(Q)}^2)^{1/2} \leq c_2 2^{-rl + \max(1/p - 1/2, 0)dl} \quad (83)$$

and, if  $\frac{r}{d} > \frac{1}{p} \vee (\frac{r}{d} = \frac{1}{p} \wedge p = 1)$ ,

$$\sup_{f \in B_{W_p^r(Q)}} \|f - P_{l,0}f\|_{L_2(Q)} \leq c_3 2^{-rl + \max(1/p - 1/2, 0)dl}. \quad (84)$$

*Proof of Theorem 6.1.* For  $l \in \mathbb{N}_0$  let  $H_l \subset L_2(Q)$  be defined by

$$H_l = \text{span} \{ \eta_{li} \pi : i \in \mathcal{I}_l, \pi \in \mathcal{P}^{d\varrho} \},$$

where  $\mathcal{P}^{d\varrho}$  denotes the space of polynomials on  $\mathbb{R}^d$  of degree  $\leq d\varrho$ . The scaled and shifted polynomials appearing in (82) all belong to this space. It follows from (82) that  $P_{l,0}$  and  $P_{l,\omega_1}$  have the form required for (57) and (58) of Lemma 5.2. Taking into account that by (81) we have  $|\mathcal{I}_l| \leq 2^{dl}$ , the upper bound for  $e_n^{\text{ran}}(W_Q, B_{W_p^r(Q)} \times \Omega, \mathbb{R})$  of (76) and that of (77) follow directly from Lemma 5.2 and Proposition 6.2. The upper bound of (78) is trivial.

Finally, the upper bound of (76) for  $e_n^{\text{det}}(W_Q, B_{W_p^r(Q)} \times \Omega, \mathbb{R})$  follows from (83) by using (59) of Lemma 5.2 with  $m = d$  and  $\Phi_{ij} : \mathbb{R}^d \rightarrow Q$ ,  $\Theta_{ij} : \mathbb{R}^d \rightarrow H_l$  ( $i \in \mathcal{I}_l, j = 1, \dots, \kappa$ ) given by

$$\begin{aligned} \Phi_{ij}(a_1, \dots, a_d) &= x_{li} + b2^{-l}(z_j + \delta(\Phi(a_1), \dots, \Phi(a_d))) \in Q \\ (\Theta_{ij}(a_1, \dots, a_d))(x) &= \eta_{li}(x) \pi_j(b^{-1}2^l(x - x_{li}) - \delta(\Phi(a_1), \dots, \Phi(a_d))) \quad (x \in Q), \end{aligned}$$

where  $\Phi : \mathbb{R} \rightarrow [0, 1]$  is the standard normal cumulative distribution function.

To show the lower bounds, let  $\psi_j$  be as in (46). It follows from the definitions (73) and (74) that there are constants  $c_1, c_2 > 0$  such that for  $m \in \mathbb{N}$ ,  $(\alpha_i)_{i=1}^{m^d} \subset \mathbb{R}$

$$c_1 m^{r-d/p} \|(\alpha_i)\|_{\ell_p^{m^d}} \leq \left\| \sum_{i=1}^{m^d} \alpha_i \psi_i \right\|_{W_p^r(Q)} \leq c_2 m^{r-d/p} \|(\alpha_i)\|_{\ell_p^{m^d}}, \quad (85)$$

hence

$$c_2^{-1} m^{-r} \Psi_m^1 \subseteq B_{W_p^r(Q)}, \quad c_2^{-1} m^{-r+d/p} \Psi_m^2 \subseteq B_{W_p^r(Q)}. \quad (86)$$

Given  $n \in \mathbb{N}$ , we set  $m = \lceil (8n)^{1/d} \rceil + 1$  and conclude from Proposition 5.1

$$e_n^{\text{ran}}(W_Q, B_{\mathcal{W}_p^r(Q)} \times \Omega, \mathbb{R}) \geq c \max(n^{-r/d}, n^{-r/d+1/p-1/2}), \quad (87)$$

showing the lower bounds in (76) and (77). Under the assumptions of (78) Theorem 4.3 of [6] states that there is a constant  $c > 0$  such that for all  $n$

$$e_n^{\text{det}}(J_p^r, B_{\mathcal{W}_p^r(Q)} \cap C(Q), L_2(Q)) \geq c.$$

Now the lower bound in (78) follows from (50). □

**Remark.** The lower bounds of Theorem 6.1 hold for arbitrary  $\Lambda_2$ , including the case of full information about  $\omega$ . The algorithms realizing the upper bounds use linear information of the stochastic process, see (55) and (56).

Based on the results above and that of Section 5 of [6], statements similar to Theorem 6.1 can also be derived for Besov and Bessel potential spaces. Let us just mention one result in this direction. Let  $1 \leq p, q \leq \infty$ ,  $r \in \mathbb{R}$ ,  $r/d > \max(1/p - 1/2, 0)$ , and let  $B_{pq}^r(Q)$  denote the Besov space. For the definition on  $\mathbb{R}^d$  see [18, 19], the case of bounded Lipschitz domains can be found in [20, 21]. Notice that for  $1 \leq p < \infty$ ,  $r \notin \mathbb{N}_0$ ,  $B_{pp}^r(Q)$  is the Slobodeckij space (see [18, 19]), considered for  $d = 1$  in [3]. Under the above conditions,  $B_{pq}^r(Q)$  is compactly embedded into  $L_2(Q)$ .

**Theorem 6.3.** *Let  $d \in \mathbb{N}$ ,  $1 \leq p, q \leq \infty$ ,  $r \in \mathbb{R}$ ,  $r/d > \max(1/p - 1/2, 0)$ . Then*

$$e_n^{\text{det}}(W_Q, B_{B_{pq}^r(Q)} \times \Omega, \mathbb{R}) \asymp e_n^{\text{ran}}(W_Q, B_{B_{pq}^r(Q)} \times \Omega, \mathbb{R}) \asymp n^{-\frac{r}{d} + \max(\frac{1}{p} - \frac{1}{2}, 0)}. \quad (88)$$

The upper bounds in the case  $d = 1$ ,  $0 < r < 2$ ,  $p = q \geq 2$  where obtained (under an additional growth condition) by Eisenmann and Kruse in [3]. Theorem 6.3 shows that these bounds are sharp.

*Proof.* The upper bounds follow from Proposition 5.1 of [6] in the same way as those of Theorem 6.1 from Proposition 6.2 above.

According to Theorem 2.3.2 of [2], there are constants  $c_1, c_2 > 0$  such that for  $m \in \mathbb{N}$ ,  $(\alpha_i)_{i=1}^{m^d} \subset \mathbb{R}$

$$c_1 m^{r-d/p} \|(\alpha_i)\|_{\ell_p^{m^d}} \leq \left\| \sum_{i=1}^{m^d} \alpha_i \psi_i \right\|_{B_{pq}^r(Q)} \leq c_2 m^{r-d/p} \|(\alpha_i)\|_{\ell_p^{m^d}}.$$

The rest of the lower bound proof is analogous to that of Theorem 6.1. □

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