Lower bounds for the number of random bits in Monte Carlo algorithms

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Abstract We continue the study of restricted Monte Carlo algorithms in a general setting. Here we show a lower bound for minimal errors in the setting with finite restriction in terms of deterministic minimal errors. This generalizes a result of [11] to the adaptive setting. As a consequence, the lower bounds on the number of random bits from [11] also hold in this setting. We also derive a lower bound on the number of needed bits for integration of Lipschitz functions over the Wiener space, complementing a result of [5].

1 Introduction

Restricted Monte Carlo algorithms were considered in [12, 13, 16, 11, 14, 3, 17, 4, 5, 6]. Restriction usually means that the algorithm has access only to random bits or to random variables with finite range. Most of these papers on restricted randomized algorithms consider the non-adaptive case. Only [5] includes adaptivity, but considers a class of algorithms where each information call is followed by one random bit call.

A general definition restricted Monte Carlo algorithms was given in [10]. It extends the previous notions in two ways: Firstly, it includes full adaptivity, and secondly, it includes models in which the algorithms have access to an arbitrary, but fixed set of random variables, for example, uniform distributions on $[0,1]$. In [10] the relation of restricted to unrestricted randomized algorithms was studied. In particular, it was shown that for each such restricted setting there is a computational problem that can be solved in the unrestricted randomized setting but not under the restriction.

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The aim of the present paper is to continue the study of the restricted setting. The main result is a lower bound for minimal errors in the setting with a finite restriction in terms of deterministic minimal errors. This generalizes a corresponding result from [11], see Proposition 1 there, to the adaptive setting with arbitrary finite restriction. The formal proof in this setting is technically more involved. As a consequence the lower bounds on the number of random bits from [11] also hold in this setting. Another corollary concerns integration of Lipschitz functions over the Wiener space [5]. It shows that the number of random bits used in the algorithm from [5] is optimal, up to logarithmic factors.

2 Restricted randomized algorithms in a general setting

We work in the framework of information-based complexity theory (IBC) [13, 15], using specifically the general approach from [7, 8]. We recall the notion of a restricted randomized algorithm as recently introduced in [10]. This section is kept general, for specific examples illustrating this setup we refer to the integration problem considered in [10] as well as to the problems studied in Section 4.

We consider an abstract numerical problem

$$\mathcal{P} = (F, G, S, K, \Lambda), \tag{1}$$

where F and K are a non-empty sets, G is a Banach space, S a mapping from F to G, and Λ a nonempty set of mappings from F to K. The operator S is understood to be the solution operator that sends the input $f \in F$ to the exact solution $S(f)$ and Λ is the set of information functionals about the input $f \in F$ that can be exploited by an algorithm.

A probability space with access restriction is a tuple

$$\mathcal{R} = \big((\Omega, \Sigma, \mathbb{P}), K', \Lambda'\big), \tag{2}$$

with $(\Omega, \Sigma, \mathbb{P})$ a probability space, K' a non-empty set, and Λ' a non-empty set of mappings from Ω to K'. Define

$$\bar{K} = K \dot{\cup} K', \quad \bar{\Lambda} = \Lambda \dot{\cup} \Lambda',$$

where $\dot{\cup}$ is the disjoint union, and for $\lambda \in \bar{\Lambda}$, $f \in F$, $\omega \in \Omega$ we set

$$\lambda(f, \omega) = \begin{cases} \lambda(f) & \text{if} \quad \lambda \in \Lambda \\ \lambda(\omega) & \text{if} \quad \lambda \in \Lambda'. \end{cases}$$

An $\mathcal{R}$-restricted randomized algorithm for problem $\mathcal{P}$ is a tuple

$$A = ((L_i)_{i=1}^{\infty}, (\tau_i)_{i=0}^{\infty}, (\varphi_i)_{i=0}^{\infty})$$

such that $L_1 \in \bar{\Lambda}$, $\tau_0 \in \{0, 1\}$, $\varphi_0 \in G$, and for $i \in \mathbb{N}$

$$L_{i+1} : \bar{K}^i \to \bar{\Lambda}, \quad \tau_i : \bar{K}^i \to \{0,1\}, \quad \varphi_i : \bar{K}^i \to G$$

are any mappings. Given $f \in F$ and $\omega \in \Omega$, we define $(\lambda_i)_{i=1}^\infty$ with $\lambda_i \in \bar{\Lambda}$ as follows:

$$\lambda_1 = L_1, \quad \lambda_i = L_i(\lambda_1(f,\omega), \dots, \lambda_{i-1}(f,\omega)) \quad (i \ge 2). \tag{3}$$

If $\tau_0 = 1$, we define

$$\mathrm{card}_{\bar{\Lambda}}(A,f,\omega) = \mathrm{card}_{\Lambda}(A,f,\omega) = \mathrm{card}_{\Lambda'}(A,f,\omega) = 0.$$

If $\tau_0 = 0$, let $\mathrm{card}_{\bar{\Lambda}}(A,f,\omega)$ be the first integer $n \ge 1$ with

$$\tau_n(\lambda_1(f,\omega), \dots, \lambda_n(f,\omega)) = 1,$$

if there is such an n. If $\tau_0 = 0$ and no such $n \in \mathbb{N}$ exists, put $\mathrm{card}_{\bar{\Lambda}}(A,f,\omega) = \infty$. Furthermore, set

$$\mathrm{card}_{\Lambda}(A,f,\omega) = |\{k \le \mathrm{card}_{\bar{\Lambda}}(A,f,\omega) : \lambda_k \in \Lambda\}|$$
$$\mathrm{card}_{\Lambda'}(A,f,\omega) = |\{k \le \mathrm{card}_{\bar{\Lambda}}(A,f,\omega) : \lambda_k \in \Lambda'\}|.$$

We have $\mathrm{card}_{\bar{\Lambda}}(A,f,\omega) = \mathrm{card}_{\Lambda}(A,f,\omega) + \mathrm{card}_{\Lambda'}(A,f,\omega)$. The output $A(f,\omega)$ of algorithm A at input (f,ω) is defined as

$$A(f,\omega) = \begin{cases} \varphi_0 & \text{if} \quad \mathrm{card}_{\bar{\Lambda}}(A,f,\omega) \in \{0,\infty\} \\ \varphi_n(\lambda_1(f,\omega), \dots, \lambda_n(f,\omega)) & \text{if} \quad 1 \le \mathrm{card}_{\bar{\Lambda}}(A,f,\omega) = n < \infty. \end{cases} \tag{4}$$

Thus, a restricted randomized algorithm can access the randomness of $(\Omega, \Sigma, \mathbb{P})$ only through the functionals $\lambda(\omega)$ for $\lambda \in \Lambda'$.

The set of all $\mathcal{R}$-restricted randomized algorithms for $\mathcal{P}$ is denoted by $\mathcal{A}^{\mathrm{ran}}(\mathcal{P},\mathcal{R})$. Let $\mathcal{A}^{\mathrm{ran}}_{\mathrm{meas}}(\mathcal{P},\mathcal{R})$ be the subset of those $A \in \mathcal{A}^{\mathrm{ran}}(\mathcal{P},\mathcal{R})$ with the following properties: For each $f \in F$ the mappings

$$\omega \to \mathrm{card}_{\Lambda}(A,f,\omega) \in \mathbb{N}_0 \cup \{\infty\}, \quad \omega \to \mathrm{card}_{\Lambda'}(A,f,\omega) \in \mathbb{N}_0 \cup \{\infty\}$$

(and hence $\omega \to \mathrm{card}_{\bar{\Lambda}}(A,f,\omega)$) are Σ-measurable and the mapping $\omega \to A(f,\omega) \in G$ is Σ-to-Borel measurable and $\mathbb{P}$-almost surely separably valued, the latter meaning that there is a separable subspace $G_f \subset G$ such that $\mathbb{P}(\{\omega \in \Omega : A(f,\omega) \in G_f\}) = 1$. The error of $A \in \mathcal{A}^{\mathrm{ran}}_{\mathrm{meas}}(\mathcal{P},\mathcal{R})$ is defined as

$$e(\mathcal{P},A) = \sup_{f \in F} \mathbb{E}\, \|S(f) - A(f,\omega)\|_G. \tag{5}$$

Given $n,k \in \mathbb{N}_0$, we define $\mathcal{A}^{\mathrm{ran}}_{n,k}(\mathcal{P},\mathcal{R})$ to be the set of those $A \in \mathcal{A}^{\mathrm{ran}}_{\mathrm{meas}}(\mathcal{P},\mathcal{R})$ satisfying for each $f \in F$

$$\mathbb{E}\,\mathrm{card}_{\Lambda}(A,f,\omega) \le n, \quad \mathbb{E}\,\mathrm{card}_{\Lambda'}(A,f,\omega) \le k.$$

The (n,k)-th minimal $\mathcal{R}$-restricted randomized error of S is defined as

$$e_{n,k}^{\mathrm{ran}}(\mathcal{P},\mathcal{R}) = \inf_{A\in\mathcal{A}_{n,k}^{\mathrm{ran}}(\mathcal{P},\mathcal{R})} e(\mathcal{P},A). \tag{6}$$

Special cases are the following: An access restriction $\mathcal{R}$ is called finite, if

$$|K'| < \infty, \quad \lambda^{-1}(\{u\}) \in \Sigma \quad (\lambda' \in \Lambda', u \in K'). \tag{7}$$

In this case any $\mathcal{R}$-restricted randomized algorithm satisfies the following. For fixed $i \in \mathbb{N}_0$ and $f \in F$ the functions (see (3))

$$\omega \to L_i(\lambda_1(f,\omega),\dots,\lambda_{i-1}(f,\omega)) \in \overline{\Lambda}, \quad \omega \to \lambda_i(f,\omega) \in \overline{K}$$

take finitely many values and are Σ-to-$\Sigma_0(\overline{\Lambda})$-measurable (respectively Σ-to-$\Sigma_0(\overline{K})$-measurable), where $\Sigma_0(M)$ denotes the σ-algebra generated by the finite subsets of a set M. This is readily checked by induction. It follows that the mapping

$$\omega \to \tau_i(\lambda_1(f,\omega),\dots,\lambda_i(f,\omega)) \in \{0,1\}$$

is measurable and

$$\omega \to \varphi_i(\lambda_1(f,\omega),\dots,\lambda_i(f,\omega)) \in G$$

takes only finitely many values and is Σ-to-Borel-measurable. Consequently, for each $f \in F$ the functions $\mathrm{card}(A,f,\omega)$ and $\mathrm{card}'(A,f,\omega)$ are Σ-measurable, $A(f,\omega)$ takes only countably many values and is Σ-to-Borel-measurable, hence $A \in \mathcal{A}_{\mathrm{meas}}^{\mathrm{ran}}(\mathcal{P},\mathcal{R})$.

An access restriction is called bit restriction, if

$$|K'| = 2, \quad \Lambda' = \{\xi_j : j \in \mathbb{N}\} \tag{8}$$

with $\xi_j : \Omega \to K' = \{u_0,u_1\}$ an independent sequence of random variables such that

$$P(\{\xi_j = u_0\}) = P(\{\xi_j = u_1\}) = 1/2, \quad (j \in \mathbb{N}). \tag{9}$$

The corresponding restricted randomized algorithms are called bit Monte Carlo algorithms. A non-adaptive version of these was considered in [11, 14, 3, 17].

Most frequently used is the case of uniform distributions on $[0,1]$. This means $K' = [0,1]$ and $\Lambda' = \{\eta_j : j \in \mathbb{N}\}$, with (η_j) being independent uniformly distributed on $[0,1]$ random variables over $(\Omega,\Sigma,\mathbb{P})$.

We also use the notion of a deterministic and of an (unrestricted) randomized algorithm and the corresponding notions of minimal errors. For this we refer to [7, 8], as well as to Section 2 of [10]. Let us however mention that the definition of a deterministic algorithm follows a similar scheme as the one given above. More than that, we can give an equivalent definition of a deterministic algorithm, viewing it as a special case of a randomized algorithm with an arbitrary restriction $\mathcal{R}$. Namely, a deterministic algorithm is an $\mathcal{R}$-restricted randomized algorithm A with

$$L_1 \in \Lambda, \quad L_{i+1}(K^i) \subseteq \Lambda \quad (i \in \mathbb{N}).$$

Consequently, for each $f \in F$ and $\omega, \omega_1 \in \Omega$ we have $\mathrm{card}_{\Lambda'}(A, f, \omega) = 0$ and

$$\begin{aligned} A(f) &:= A(f, \omega) = A(f, \omega_1) \\ \mathrm{card}(A, f) &:= \mathrm{card}_{\bar{\Lambda}}(A, f, \omega) = \mathrm{card}_{\Lambda}(A, f, \omega) = \mathrm{card}_{\Lambda}(A, f, \omega_1). \end{aligned}$$

Thus, such an algorithm ignores $\mathcal{R}$ completely. For a deterministic algorithm A relation (5) turns into

$$e(\mathcal{P}, A) = \sup_{f \in F} \|S(f) - A(f)\|_G. \tag{10}$$

A deterministic algorithm is in $\mathcal{A}_{n,k}^{\mathrm{ran}}(\mathcal{P}, \mathcal{R})$ iff $\sup_{f \in F} \mathrm{card}(A, f) \le n$. Taking the infimum in (6) over all such A gives the n-th minimal error in the deterministic setting $e_n^{\mathrm{det}}(\mathcal{P})$. Clearly, $e(\mathcal{P}, A)$ and $e_n^{\mathrm{det}}(\mathcal{P})$ do not depend on $\mathcal{R}$. It follows that for each restriction $\mathcal{R}$ and $n, k \in \mathbb{N}_0$

$$e_{n,k}^{\mathrm{ran}}(\mathcal{P}, \mathcal{R}) \le e_n^{\mathrm{det}}(\mathcal{P}).$$

A restricted randomized algorithm is a special case of an (unrestricted) randomized algorithm. Being intuitively clear, this was formally checked in [10], Proposition 2.1 and Corollary 2.2. Moreover, it was shown there that for each restriction $\mathcal{R}$ and $n, k \in \mathbb{N}_0$

$$e_n^{\mathrm{ran}}(\mathcal{P}) \le e_{n,k}^{\mathrm{ran}}(\mathcal{P}, \mathcal{R}),$$

where $e_n^{\mathrm{ran}}(\mathcal{P})$ denotes the n-th minimal error in the randomized setting,

3 Deterministic vs. Restricted Randomized Algorithms

In this section we derive a relation between minimal restricted randomized errors and minimal deterministic errors for general problems. Variants of the following result have been obtained for non-adaptive random bit algorithms in [11, Prop. 1], and for adaptive algorithms that ask for random bits and function values in alternating order in [5]. Obviously, the latter does not permit to analyze a trade-off between the number of random bits and the number of function values to be used in a computation.

Theorem 1 *For all problems $\mathcal{P} = (F, G, S, K, \Lambda)$ and probability spaces with finite access restriction $\mathcal{R} = \big((\Omega, \Sigma, \mathbb{P}), K', \Lambda'\big)$, see* (7)*, and for all $n, k \in \mathbb{N}_0$ we have*

$$e_{n,k}^{\mathrm{ran}}(\mathcal{P}, \mathcal{R}) \ge \frac{1}{3} e_{3n|K'|^{3k}}^{\mathrm{det}}(\mathcal{P}).$$

Without loss of generality in the sequel we only consider access restrictions with the property $K \cap K' = \emptyset$, thus $\bar{K} = K \cup K'$, $\bar{\Lambda} = \Lambda \cup \Lambda'$.

Lemma 1 *Let $n, k \in \mathbb{N}_0$, let A be a randomized algorithm for $\mathcal{P}$ with access restriction $\mathcal{R} = \big((\Omega, \Sigma, \mathbb{P}), K', \Lambda'\big)$. For each $f \in F$ let*

$$B_f = \{\omega \in \Omega\colon \operatorname{card}(A, f, \omega) \le n,\ \operatorname{card}'(A, f, \omega) \le k\}. \tag{11}$$

Then there is an $\mathcal{R}$-restricted randomized algorithm $\tilde{A}$ for $\tilde{\mathcal{P}} = (F, \tilde{G}, \tilde{S}, \Lambda, K)$, where $\tilde{G} = G \oplus \mathbb{R}$ and $\tilde{S} = (S(f), 0)$, satisfying for all $f \in F$ and $\omega \in \Omega$

$$\operatorname{card}(\tilde{A}, f, \omega) \le n \tag{12}$$
$$\operatorname{card}'(\tilde{A}, f, \omega) \le k \tag{13}$$
$$\tilde{A}(f, \omega) = (A(f, \omega) \cdot 1_{B_f}(\omega), 1_{B_f}(\omega)). \tag{14}$$

Proof Let $A = ((L_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty)$. For $i \in \mathbb{N}_0$ and $a = (a_1, \dots, a_i) \in \overline{K}^i$ let

$$\begin{aligned} d_{i+1}(a) &= |\{L_1, L_2(a_1), \dots, L_{i+1}(a_1, \dots, a_i)\} \cap \Lambda| \\ d'_{i+1}(a) &= |\{L_1, L_2(a_1), \dots, L_{i+1}(a_1, \dots, a_i)\} \cap \Lambda'| \\ \zeta_i(a) &= \begin{cases} 1 \text{ if } (d_{i+1}(a) > n) \vee (d'_{i+1}(a) > k) \\ 0 \quad \text{otherwise.} \end{cases} \end{aligned}$$

Now we define $\tilde{A} = ((L_i)_{i=1}^\infty, (\tilde{\tau}_i)_{i=0}^\infty, (\tilde{\varphi}_i)_{i=0}^\infty)$ by setting for $i \in \mathbb{N}_0$ and $a \in \overline{K}^i$

$$\begin{aligned} \tilde{\tau}_i(a) &= \max(\tau_i(a), \zeta_i(a)) \\ \tilde{\varphi}_i(a) &= \begin{cases} (\varphi_i(a), 1) & \text{if} \quad \zeta_i(a) \le \tau_i(a) \\ (0, 0) & \text{if} \quad \zeta_i(a) > \tau_i(a). \end{cases} \end{aligned}$$

To show (12)–(14) we fix $f \in F$, $\omega \in \Omega$ and define

$$a_1 = L_1(f, \omega), \qquad a_i = (L_i(a_1, \dots, a_{i-1}))(f, \omega) \quad (i \ge 2).$$

Let $m = \overline{\operatorname{card}}(A, f, \omega)$ and let q be the smallest number $q \in \mathbb{N}_0$ with $\zeta_q(a_1, \dots, a_q) = 1$. First assume that $\omega \in B_f$. Then for all $i < m$

$$(d_{i+1}(a_1, \dots, a_i) \le n) \wedge (d'_{i+1}(a_1, \dots, a_i) \le k),$$

thus $\zeta_i(a_1, \dots, a_i) = 0$ and therefore $\tilde{\tau}_i(a_1, \dots, a_i) = 0$. Furthermore,

$$\zeta_i(a_1, \dots, a_m) \le \tau_m(a_1, \dots, a_m) = 1,$$

which means $\overline{\operatorname{card}}(\tilde{A}, f, \omega) = m$,

$$\begin{aligned} \operatorname{card}(\tilde{A}, f, \omega) &= d_m(a_1, \dots, a_{m-1}) \le n \\ \operatorname{card}'(\tilde{A}, f, \omega) &= d'_m(a_1, \dots, a_{m-1}) \le k \\ \tilde{A}(f, \omega) &= (\varphi_m(a_1, \dots, a_m), 1) = (A(f, \omega), 1). \end{aligned}$$

Now let $\omega \in \Omega \setminus B_f$, hence

$$\tau_0 = \tau_1(a_1) = \cdots = \tau_q(a_1, \ldots, a_q) = 0$$
$$(d_{q+1}(a_1, \ldots, a_q) > n) \vee (d'_{q+1}(a_1, \ldots, a_q) > k),$$

thus $\tilde{\tau}_q(a_1, \ldots, a_q) = 1$. Consequently,

$$\begin{aligned} \operatorname{card}(\tilde{A}, f, \omega) &\le d_q(a_1, \ldots, a_{q-1}) \le n \\ \operatorname{card}'(\tilde{A}, f, \omega) &\le d'_q(a_1, \ldots, a_{q-1}) \le k \\ \tilde{A}(f, \omega) &= (0, 0). \end{aligned}$$

□

The key ingredient of the proof of Theorem 1 is the following

Lemma 2 *Let $n, k \in \mathbb{N}_0$ and let A be a randomized algorithm for $\mathcal{P}$ with finite access restriction $\mathcal{R} = \big((\Omega, \Sigma, \mathbb{P}), K', \Lambda'\big)$ such that*

$$\operatorname{card}(A, f, \omega) \le n, \quad \operatorname{card}'(A, f, \omega) \le k \tag{15}$$

for all $f \in F$ and $\omega \in \Omega$. Then there exists a deterministic algorithm A^* for $\mathcal{P}$ with

$$A^*(f) = \mathbb{E}\,(A(f, \cdot)), \quad \operatorname{card}(A^*, f) \le n|K'|^k \quad (f \in F). \tag{16}$$

Proof Let $\mathcal{P} = (F, G, S, K, \Lambda)$, $A = ((L_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty)$. We argue by induction over $m = n + k$. If $m = 0$, then $\tau_0 = 1$, hence $\overline{\operatorname{card}}(A, f, \omega) = 0$, thus $A(f, \omega) = \varphi_0$ for all $f \in F$ and $\omega \in \Omega$, and the result follows.

Now let $m \ge 1$. We can assume that $\tau_0 = 0$, otherwise A satisfies (15) with $n = k = 0$ and we are back to the case $m = 0$. Let $\tilde{K} \subset \overline{K}$ be defined by

$$\tilde{K} = \begin{cases} \big\{u \in K : \ L_1^{-1}(\{u\}) \ne \emptyset\big\} & \text{if} \quad L_1 \in \Lambda \\ \big\{u \in K' : \ \mathbb{P}(L_1^{-1}(\{u\})) \ne 0\big\} & \text{if} \quad L_1 \in \Lambda'. \end{cases}$$

For every $u \in \tilde{K}$ we define a problem $\mathcal{P}_u = (F_u, G, S_u, K, \Lambda_u)$ and a probability space with access restriction $\mathcal{R}_u = \big((\Omega_u, \Sigma_u, \mathbb{P}_u), K', \Lambda'_u\big)$ as follows. If $L_1 \in \Lambda$, we set $\mathcal{R}_u = \mathcal{R}$ and

$$F_u = \{f \in F : \ L_1(f) = u\}, \quad S_u = S|_{F_u}, \quad \Lambda_u = \{\lambda|_{F_u} : \ \lambda \in \Lambda\}.$$

If $L_1 \in \Lambda'$, we put $\mathcal{P}_u = \mathcal{P}$ and

$$\Omega_u = \{\omega \in \Omega : \ L_1(\omega) = u\}, \quad \Sigma_u = \{B \cap \Omega_u : \ B \in \Sigma\}$$
$$\mathbb{P}_u(C) = \mathbb{P}\,(\Omega_u)^{-1}\,\mathbb{P}\,(C) \quad (C \in \Sigma_u), \quad \Lambda'_u = \{\lambda'|_{\Omega_u} : \ \lambda \in \Lambda'\}.$$

Let $\varrho_u : \Lambda \cup \Lambda' \to \Lambda_u \cup \Lambda'_u$ be defined as

$$\varrho_u(\lambda) = \begin{cases} \lambda|_{F_u} & \text{if} \quad \lambda \in \Lambda \\ \lambda|_{\Omega_u} & \text{if} \quad \lambda \in \Lambda' \end{cases}$$

and let $\sigma_u : \Lambda_u \cup \Lambda'_u \to \Lambda \cup \Lambda'$ be any mapping satisfying

$$\varrho_u \circ \sigma_u = \mathrm{id}_{\Lambda_u \cup \Lambda'_u}. \tag{17}$$

Furthermore, we define a random algorithm $A_u = ((L_{i,u})_{i=1}^{\infty}, (\tau_{i,u})_{i=0}^{\infty}, (\varphi_{i,u})_{i=0}^{\infty})$ for $\mathcal{P}_u$ with access restriction $\mathcal{R}_u$ by setting for $i \geq 0$, $z_1, \dots, z_i \in \overline{K}$

$$L_{i+1,u}(z_1, \dots, z_i) = \big(\varrho_u \circ L_{i+2}\big)(u, z_1, \dots, z_i) \tag{18}$$

$$\tau_{i,u}(z_1, \dots, z_i) = \tau_{i+1}(u, z_1, \dots, z_i) \tag{19}$$

$$\varphi_{i,u}(z_1, \dots, z_i) = \varphi_{i+1}(u, z_1, \dots, z_i) \tag{20}$$

(in this and similar situations below the case $i = 0$ with variables $z_1, \dots, z_i$ is understood in the obvious way: no dependence on $z_1, \dots, z_i$).

Next we establish the relation of the algorithms A_u to A. Fix $f \in F_u$, $\omega \in \Omega_u$, and let $(a_i)_{i=1}^{\infty} \subseteq \overline{K}$ be given by

$$a_1 = L_1(f, \omega) = u \tag{21}$$

$$a_i = \big(L_i(a_1, \dots, a_{i-1})\big)(f, \omega) \quad (i \geq 2), \tag{22}$$

and similarly $(a_{i,u})_{i=1}^{\infty} \subseteq \overline{K}$ by

$$a_{i,u} = \big(L_{i,u}(a_{1,u}, \dots, a_{i-1,u})\big)(f, \omega). \tag{23}$$

We show by induction that

$$a_{i,u} = a_{i+1} \quad (i \in \mathbb{N}). \tag{24}$$

Let $i = 1$. Then (23), (18), (21), and (22) imply

$$a_{1,u} = L_{1,u}(f, \omega) = \big(L_2(u)\big)(f, \omega) = \big(L_2(a_1)\big)(f, \omega) = a_2.$$

For the induction step we let $j \in \mathbb{N}$ and suppose that (24) holds for all $i \leq j$. Then (23), (18), (24), and (22) yield

$$\begin{aligned} a_{j+1,u} &= (L_{j+1,u}(a_{1,u}, \dots, a_{j,u}))(f, \omega) = (L_{j+2}(u, a_{1,u}, \dots, a_{j,u}))(f, \omega) \\ &= (L_{j+2}(a_1, a_2, \dots, a_{j+1}))(f, \omega) = a_{j+2}. \end{aligned}$$

This proves (24). As a consequence of this relation and of (18), (19), and (20) we obtain for all $i \in \mathbb{N}_0$

$$\begin{aligned} L_{i+1,u}(a_{1,u}, \dots, a_{i,u}) &= \big(\varrho_u \circ L_{i+2}\big)(u, a_{1,u}, \dots, a_{i,u}) = \big(\varrho_u \circ L_{i+2}\big)(a_1, \dots, a_{i+1}) \\ \tau_{i,u}(a_{1,u}, \dots, a_{i,u}) &= \tau_{i+1}(u, a_{1,u}, \dots, a_{i,u}) = \tau_{i+1}(a_1, \dots, a_{i+1}) \\ \varphi_{i,u}(a_{1,u}, \dots, a_{i,u}) &= \varphi_{i+1}(u, a_{1,u}, \dots, a_{i,u}) = \varphi_{i+1}(a_1, \dots, a_{i+1}). \end{aligned}$$

Hence, for all $f \in F_u$ and $\omega \in \Omega_u$

$$\begin{aligned} \overline{\mathrm{card}}(A_u, f, \omega) &= \overline{\mathrm{card}}(A, f, \omega) - 1 \\ A_u(f, \omega) &= A(f, \omega). \end{aligned} \tag{25}$$

Furthermore, if $L_1 \in \Lambda$, then

$$\begin{aligned} \mathrm{card}(A_u, f, \omega) &= \mathrm{card}(A, f, \omega) - 1 \le n - 1 \\ \mathrm{card}'(A_u, f, \omega) &= \mathrm{card}'(A, f, \omega) \le k, \end{aligned}$$

and if $L_1 \in \Lambda'$,

$$\begin{aligned} \mathrm{card}(A_u, f, \omega) &= \mathrm{card}(A, f, \omega) \le n \\ \mathrm{card}'(A_u, f, \omega) &= \mathrm{card}'(A, f, \omega) - 1 \le k - 1. \end{aligned}$$

Now we apply the induction assumption and obtain a deterministic algorithm

$$A_u^* = ((L_{i,u}^*)_{i=1}^\infty, (\tau_{i,u}^*)_{i=0}^\infty, (\varphi_{i,u}^*)_{i=0}^\infty)$$

for $\mathcal{P}_u$ with

$$A_u^*(f) = \mathbb{E}_{\mathbb{P}_u}(A_u(f, \cdot)) \tag{26}$$

and

$$\mathrm{card}(A_u^*, f) \le \begin{cases} (n-1)|K'|^k & \text{if} \quad L_1 \in \Lambda \\ n|K'|^{k-1} & \text{if} \quad L_1 \in \Lambda' \end{cases} \tag{27}$$

for every $f \in F_u$.

Finally we use the algorithms A_u^* to compose a deterministic algorithm

$$A^* = ((L_i^*)_{i=1}^\infty, (\tau_i^*)_{i=0}^\infty, (\varphi_i^*)_{i=0}^\infty)$$

for $\mathcal{P}$. This and the completion of the proof is done separately for each of the cases $L_1 \in \Lambda$ and $L_1 \in \Lambda'$.

If $L_1 \in \Lambda$, then we set

$$L_1^* = L_1, \quad \tau_0^* = \tau_0 = 0, \quad \varphi_0^* = \varphi_0,$$

furthermore, for $i \in \mathbb{N}$, $z_1 \in \tilde{K}$, $z_2, \ldots, z_i \in \overline{K}$ we let (with σ_{z_1} defined by (17))

$$L_{i+1}^*(z_1, \ldots, z_i) = \left(\sigma_{z_1} \circ L_{i,z_1}^*\right)(z_2, \ldots, z_i) \tag{28}$$

$$\tau_i^*(z_1, \ldots, z_i) = \tau_{i-1,z_1}^*(z_2, \ldots, z_i) \tag{29}$$

$$\varphi_i^*(z_1, \ldots, z_i) = \varphi_{i-1,z_1}^*(z_2, \ldots, z_i). \tag{30}$$

For $i \ge 1$, $z_1 \in \overline{K} \setminus \tilde{K}$, and $z_2, \ldots, z_i \in \overline{K}$ we define

$$L_{i+1}^*(z_1, \ldots, z_i) = L_1, \quad \tau_i^*(z_1, \ldots, z_i) = 1, \quad \varphi_i^*(z_1, \ldots, z_i) = \varphi_0.$$

Let $u \in \tilde{K}$ and $f \in F_u$. We show that

$$A^*(f) = A_u^*(f) \tag{31}$$
$$\operatorname{card}(A^*, f) = \operatorname{card}(A_u^*, f) + 1. \tag{32}$$

Let $(b_i)_{i=1}^\infty \subseteq \overline{K}$ be given by

$$b_1 = L_1^*(f) = L_1(f) = u \tag{33}$$
$$b_i = \big(L_i^*(b_1, \dots, b_{i-1})\big)(f) \quad (i \ge 2), \tag{34}$$

and similarly $(b_{i,u})_{i=1}^\infty \subseteq \overline{K}$ by

$$b_{i,u} = \big(L_{i,u}^*(b_{1,u}, \dots, b_{i-1,u})\big)(f). \tag{35}$$

Then

$$b_{i+1} = b_{i,u} \quad (i \in \mathbb{N}). \tag{36}$$

Indeed, for $i = 1$ we conclude from (34), (33), (28), and (35)

$$b_2 = (L_2^*(b_1))(f) = (L_2^*(u))(f) = L_{1,u}^*(f) = b_{1,u}.$$

Now let $j \in \mathbb{N}$ and assume (36) holds for all $i \le j$. By (34), (33), (28), and (35)

$$\begin{aligned} b_{j+2} &= (L_{j+2}^*(b_1, b_2, \dots, b_{j+1}))(f) = (L_{j+2}^*(u, b_{1,u}, \dots, b_{j,u}))(f) \\ &= (L_{j+1,u}^*(b_{1,u}, \dots, b_{j,u}))(f) = b_{j+1,u}. \end{aligned}$$

This proves (36). It follows from (36), (33), (29), and (30) that for all $i \in \mathbb{N}_0$

$$\begin{aligned} \tau_{i+1}^*(b_1, \dots, b_{i+1}) &= \tau_{i+1}^*(u, b_{1,u}, \dots, b_{i,u}) = \tau_{i,u}^*(b_{1,u}, \dots, b_{i,u}) \\ \varphi_{i+1}^*(b_1, \dots, b_{i+1}) &= \varphi_{i+1}^*(u, b_{1,u}, \dots, b_{i,u}) = \varphi_{i,u}^*(b_{1,u}, \dots, b_{i,u}). \end{aligned}$$

This shows (31) and (32). From (31), (26), and (25) we conclude for $u \in \tilde{K}$, $f \in F_u$, recalling that $\mathcal{R}_u = \mathcal{R}$,

$$A^*(f) = A_u^*(f) = \mathbb{E}_{\mathbb{P}}(A_u(f, \cdot)) = \mathbb{E}_{\mathbb{P}}(A(f, \cdot)).$$

Since $\cup_{u \in \tilde{K}} F_u = F$, the first relation of (16) follows. The second relation is a direct consequence of (32) and (27), completing the induction for the case $L_1 \in \Lambda$.

If $L_1 \in \Lambda'$, then we use the algorithms $(A_u^*)_{u \in \tilde{K}}$ for $\mathcal{P}_u = \mathcal{P}$ and Lemma 3 of [8] to obtain a deterministic algorithm A^* for $\mathcal{P}$ such that for $f \in F$

$$A^*(f) = \sum_{u \in \tilde{K}} \mathbb{P}(L_1^{-1}(\{u\}) A_u^*(f) \tag{37}$$
$$\operatorname{card}(A^*, f) = \sum_{u \in \tilde{K}} \operatorname{card}(A_u^*, f). \tag{38}$$

It follows from (37), (26), and (25) that

$$\begin{aligned} A^*(f) &= \sum_{u\in K'\colon \mathbb{P}(L_1^{-1}(\{u\}))>0} \mathbb{P}(L_1^{-1}(\{u\})\mathbb{E}_{\mathbb{P}_u} A_u(f,\cdot) \\ &= \sum_{u\in K'\colon \mathbb{P}(L_1^{-1}(\{u\}))>0} \int_{L_1^{-1}(\{u\})} A_u(f,\omega)d\mathbb{P}(\omega) \\ &= \sum_{u\in K'\colon \mathbb{P}(L_1^{-1}(\{u\}))>0} \int_{L_1^{-1}(\{u\})} A(f,\omega)d\mathbb{P}(\omega) = \mathbb{E}_{\mathbb{P}} A_u(f,\cdot). \end{aligned}$$

Furthermore, (27) and (38) imply $\operatorname{card}(A^*, f) \le n|K'|^k$. □

Proof of Theorem 1 The proof is similar to the proof of [5, Lem. 11]. Let $\delta > 0$ and let

$$A = ((L_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty) \in \mathcal{A}_{n,k}^{\text{ran}}(\mathcal{P}, \mathcal{R})$$

be a randomized algorithm for $\mathcal{P}$ with restriction $\mathcal{R}$ satisfying

$$e(A, \mathcal{P}) \le e_{n,k}^{\text{ran}}(\mathcal{P}, \mathcal{R}) + \delta. \tag{39}$$

For $f \in F$ define

$$B_f = \{\omega \in \Omega\colon \operatorname{card}(A, f, \omega) \le 3n,\ \operatorname{card}'(A, f, \omega) \le 3k\}.$$

Observe that $B_f \in \Sigma$ and $P(B_f) \ge 1/3$. For the conditional expectation

$$\mathbb{E}\,(A(f,\cdot)\,|\,B_f) = \frac{\mathbb{E}\,\left(A(f,\cdot)\cdot 1_{B_f}\right)}{P(B_f)}$$

of $A(f,\cdot)$ given B_f we obtain

$$\begin{aligned} &3\mathbb{E}\,\|S(f) - A(f,\cdot)\|_G \\ &\ge \mathbb{E}\,\left(\|S(f) - A(f,\cdot)\|_G\,|\,B_f\right) \ge \left\|S(f) - \mathbb{E}\,\left(A(f,\cdot)\,|\,B_f\right)\right\|_G \end{aligned} \tag{40}$$

by means of Jensen's inequality. Our goal is now to design a deterministic algorithm with input-output mapping $f \mapsto \mathbb{E}\,(A(f,\cdot)\,|\,B_f)$.

From Lemma 1 we conclude that there is an $\mathcal{R}$-restricted randomized algorithm $\tilde{A} = ((L_i)_{i=1}^\infty, (\tilde{\tau}_i)_{i=0}^\infty, (\tilde{\varphi}_i)_{i=0}^\infty)$ for $\tilde{\mathcal{P}} = (F, \tilde{G}, \tilde{S}, \Lambda, K)$, where $\tilde{G} = G \oplus \mathbb{R}$ and $\tilde{S}(f) = (S(f), 0)$ $(f \in F)$, satisfying for all $f \in F$ and $\omega \in \Omega$

$$\begin{aligned} &\operatorname{card}(\tilde{A}, f, \omega) \le 3n, \quad \operatorname{card}'(\tilde{A}, f, \omega) \le 3k, \\ &\tilde{A}(f,\omega) = (A(f,\omega)\cdot 1_{B_f}(\omega), 1_{B_f}(\omega)). \end{aligned}$$

By Lemma 2 there is a deterministic algorithm $A^* = ((L_i^*)_{i=1}^\infty, (\tau_i^*)_{i=0}^\infty, (\varphi_i^*)_{i=0}^\infty)$ for $\tilde{\mathcal{P}}$ such that for all $f \in F$

$$\operatorname{card}(A^*, f) \le 3n|K'|^{3k}, \quad A^*(f) = \left(\int_{B_f} A(f,\omega) d\mathbb{P}(\omega), \mathbb{P}(B_f) \right).$$

It remains to modify A^* as follows

$$\tilde{A}^* = ((L_i^*)_{i=1}^\infty, (\tau_i^*)_{i=0}^\infty, (\psi_i^*)_{i=0}^\infty),$$

where for $i \in \mathbb{N}_0$ and $a \in K^i$

$$\psi_i^*(a) = \begin{cases} \frac{\varphi_{i,1}^*(a)}{\varphi_{i,2}^*(a)} & \text{if} \quad \varphi_{i,2}^*(a) \neq 0 \\ 0 & \text{if} \quad \varphi_{i,2}^*(a) = 0, \end{cases}$$

with $\varphi_i^*(a) = (\varphi_{i,1}^*(a), \varphi_{i,2}^*(a))$ being the splitting into the G and $\mathbb{R}$ component. Hence for each $f \in F$

$$\begin{aligned} \operatorname{card}(\tilde{A}^*, f) &\le 3n|K'|^{3k} \\ \tilde{A}^*(f) &= \mathbb{E}\,(A(f,\cdot)\,|\,B_f), \end{aligned}$$

and therefore we conclude, using (39) and (40),

$$e_{3n|K'|^{3k}}^{\mathrm{det}}(\mathcal{P}) \le e(\tilde{A}^*, \tilde{\mathcal{P}}) \le 3e(A,\mathcal{P}) \le 3(e_{n,k}^{\mathrm{ran}}(\mathcal{P},\mathcal{R}) + \delta)$$

for each $\delta > 0$. ◻

4 Applications

4.1 Integration of functions in Sobolev spaces

Let $r, d \in \mathbb{N}$, $1 \le p < \infty$, $Q = [0,1]^d$, let $C(Q)$ be the space of continuous functions on Q, and $W_p^r(Q)$ the Sobolev space, see [1]. Then $W_p^r(Q)$ is embedded into $C(Q)$ iff

$$(p = 1 \text{ and } \quad r/d \ge 1) \quad \text{or} \quad (1 < p < \infty \text{ and } \quad r/d > 1/p). \tag{41}$$

Let $B_{W_p^r(Q)}$ be the unit ball of $W_p^r(Q)$, $B_{W_p^r(Q)} \cap C(Q)$ the set of those elements of the unit ball which are continuous (more precisely, of equivalence classes, which contain a continuous representative), and define

$$F_1 = \begin{cases} B_{W_p^r(Q)} & \text{if the embedding condition (41) holds} \\ B_{W_p^r(Q)} \cap C(Q) & \text{otherwise.} \end{cases}$$

Moreover, let $I_1 : W_p^r(Q) \to \mathbb{R}$ be the integration operator

$$I_1 f = \int_Q f(x)dx.$$

and let $\Lambda_1 = \{\delta_x : x \in Q\}$ be the set of point evaluations, where $\delta_x(f) = f(x)$. Put into the general framework of (1), we consider the problem $\mathcal{P}_1 = (F_1, \mathbb{R}, I_1, \mathbb{R}, \Lambda_1)$. Set $\bar{p} = \min(p, 2)$. Then the following is known (for (42–44) below see [9] and references therein). There are constants $c_{1-6} > 0$ such that for all $n \in \mathbb{N}_0$

$$c_1 n^{-r/d-1+1/\bar{p}} \le e_n^{\mathrm{ran}}(\mathcal{P}_1) \le c_2 n^{-r/d-1+1/\bar{p}}, \tag{42}$$

moreover, if the embedding condition holds, then

$$c_3 n^{-r/d} \le e_n^{\det}(\mathcal{P}_1) \le c_4 n^{-r/d}, \tag{43}$$

while if the embedding condition does not hold, then

$$c_5 \le e_n^{\det}(\mathcal{P}_1) \le c_6. \tag{44}$$

Theorem 1 immediately gives (compare this with the rate in the unrestricted setting (42))

Corollary 1 *Assume that the embedding condition* (41) *does not hold and let $\mathcal{R}$ be any finite access restriction, see* (7)*. Then there is a constant $c > 0$ such that for all $n, k \in \mathbb{N}$*

$$e_{n,k}^{\mathrm{ran}}(\mathcal{P}_1, \mathcal{R}) \ge c.$$

It was shown in [11], that if the embedding condition holds, then $(2 + d)\log_2 n$ random bits suffice to reach the rate of the unrestricted randomized setting, thus, if $\mathcal{R}$ is a bit restriction (see (8)–(9)), then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$

$$c_1 n^{-r/d-1+1/\bar{p}} \le e_n^{\mathrm{ran}}(\mathcal{P}_1) \le e_{n,(2+d)\log_2 n}^{\mathrm{ran}}(\mathcal{P}_1, \mathcal{R}) \le c_2 n^{-r/d-1+1/\bar{p}}. \tag{45}$$

The following consequence of Theorem 1 shows that the number of random bits used in the (non-adaptive) algorithm from [11] giving (45) is optimal up to a constant factor, also for adaptive algorithms.

Corollary 2 *Assume that the embedding condition holds and let $\mathcal{R}$ be any finite access restriction. Then for each σ with $0 < \sigma \le 1 - 1/\bar{p}$ and each $c_0 > 0$ there are constants $c_1 > 0$, $c_2 \in \mathbb{R}$ such that for all $n, k \in \mathbb{N}$*

$$e_{n,k}^{\mathrm{ran}}(\mathcal{P}_1, \mathcal{R}) \le c_0 n^{-r/d-\sigma}.$$

implies

$$k \ge c_1 \sigma \log_2 n + c_2.$$

Proof Let $\mathcal{R} = \big((\Omega, \Sigma, \mathbb{P}), K', \Lambda'\big)$. By Theorem 1 and (43),

$$c_0 n^{-r/d-\sigma} \ge e_{n,k}^{\mathrm{ran}}(\mathcal{P}_1, \mathcal{R}) \ge 3^{-1} e_{3n|K'|^{3k}}^{\det}(\mathcal{P}_1) \ge 3^{-1} c_3 (n|K'|^{3k})^{-r/d},$$

implying

$$\log_2 c_0 - \sigma \log_2 n \geq \log_2(c_3/3) - \frac{3kr}{d} \log_2 |K'|,$$

thus,

$$k \geq \frac{d}{3r \log_2 |K'|} (\sigma \log_2 n - \log_2 c_0 + \log_2(c_3/3)). \qquad \square$$

4.2 Integration of Lipschitz functions over the Wiener space

Let μ be the Wiener measure on $C([0,1])$,

$$F_2 = \{f : C([0,1]) \to \mathbb{R},\ |f(x) - f(y)| \leq \|x - y\|_{C([0,1])} \quad (x, y \in C([0,1]))\},$$

$G = \mathbb{R}$, let $I_2 : F \to \mathbb{R}$ be the integration operator given by

$$I_2 f = \int_{C([0,1])} f(x) d\mu(x),$$

and $\Lambda_2 = \{\delta_x : x \in C([0,1])\}$, so we consider the problem $\mathcal{P}_2 = (F_2, \mathbb{R}, I_2, \mathbb{R}, \Lambda_2)$. There exist constants $c_{1-4} > 0$ such that

$$c_1 n^{-1/2} (\log_2 n)^{-3/2} \leq e_n^{\text{ran}}(\mathcal{P}_2) \leq c_2 n^{-1/2} (\log_2 n)^{-1/2} \tag{46}$$

and

$$c_3 (\log_2 n)^{-1/2} \leq e_n^{\text{det}}(\mathcal{P}_2) \leq c_4 (\log_2 n)^{-1/2} \tag{47}$$

for every $n \geq 2$, see [2], Theorem 1 and Proposition 3 for (47) and Theorems 11 and 12 for (46). Moreover, it is shown in [5], Theorem 8 and Remark 9, that if $\mathcal{R}$ is a bit restriction, then there exist a constants $c_1 > 0$, $c_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq 3$

$$e_{n,\kappa(n)}^{\text{ran}}(\mathcal{P}_2, \mathcal{R}) \leq c_1 n^{-1/2} (\log_2 n)^{3/2}, \tag{48}$$

where

$$\kappa(n) = c_2 \lceil n (\log_2 n)^{-1} \log_2(\log_2 n) \rceil. \tag{49}$$

Our results imply that the number of random bits (49) used in the algorithm of [5] giving the upper bound in (48) is optimal (up to log terms) in the following sense.

Corollary 3 *Let $\mathcal{R}$ be a finite access restriction. For each $\alpha \in \mathbb{R}$ and each $c_0 > 0$ there are constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that for all $n, k \in \mathbb{N}$ with $n \geq 2$*

$$e_{n,k}^{\text{ran}}(\mathcal{P}_2, \mathcal{R}) \leq c_0 n^{-1/2} (\log_2 n)^{\alpha}.$$

implies

$$k \geq c_1 n (\log_2 n)^{-2\alpha} + c_2. \tag{50}$$

Proof Let $\mathcal{R} = \big((\Omega, \Sigma, \mathbb{P}), K', \Lambda'\big)$. We use Theorem 1 again. From (47) we obtain

$$c_0 n^{-1/2}(\log_2 n)^\alpha \ge e^{\mathrm{ran}}_{n,k}(\mathcal{P}_2, \mathcal{R}) \ge 3^{-1} e^{\mathrm{det}}_{3n|K'|^{3k}}(\mathcal{P}_2) \ge 3^{-1} c_3 \log_2(3n|K'|^{3k})^{-1/2},$$

thus

$$\log_2(3n) + 3k \log_2 |K'| \ge \frac{c_3^2}{9c_0^2} n(\log_2 n)^{-2\alpha},$$

which implies

$$k \ge (3 \log_2 |K'|)^{-1} \left(\frac{c_3^2}{9c_0^2} n(\log_2 n)^{-2\alpha} - \log_2(3n) \right). \tag{51}$$

Choosing $n_0 \in \mathbb{N}$ in such a way that for $n \ge n_0$

$$\frac{c_3^2}{18c_0^2} n(\log_2 n)^{-2\alpha} \ge \log_2(3n)$$

leads to

$$k \ge (3 \log_2 |K'|)^{-1} \left(\frac{c_3^2}{18c_0^2} n(\log_2 n)^{-2\alpha} - \log_2(3n_0) \right). \qquad \square$$

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