Lower Complexity Bounds for Parametric Stochastic Itô Integration

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Abstract

We study the complexity of pathwise approximation of parameter dependent stochastic Itô integration for C^r functions, with $r \in \mathbb{R}$, $r > 0$. Both definite and indefinite integration are considered. This complements previous results [2] for classes of functions with dominating mixed smoothness. Upper bounds are obtained by embedding of function classes and applying some generalizations of these previous results. The emphasis of the present paper lies on lower bounds. While in [2] only nonadaptive deterministic algorithms were considered, we prove here lower bounds for adaptive deterministic and randomized algorithms, both for the classes considered here as for those from [2].

1 Introduction

The complexity of stochastic integration of real-valued non-parametric functions was investigated in [15], [8], [12], [13]. In [2] the complexity of definite and indefinite stochastic Itô integration of parameter dependent random functions was studied. Classes of functions with smoothness of dominating mixed type $C^{r,\varrho}$ with integer degree of differentiability r were considered there. A multilevel Euler-Maruyama scheme was developed and analyzed to obtain the upper bounds. Moreover, matching lower bounds were shown in the deterministic nonadaptive setting. The present paper extends and complements these results in a number of respects.

First of all, we study standard isotropic C^r-smoothness. This allows to compare the results with previous ones for (non-stochastic) parametric integration obtained in [5], [3], [4], and also in [1]. However, we consider real-valued r, thus differentiable functions which satisfy suitable Hölder conditions. We discuss the extension of the results of [2] to fractional indices of smoothness. Then we derive upper bounds for C^r classes by studying their embedding into suitable $C^{r_1,\varrho}$ classes and applying the algorithm and its analysis from [2].

The main results of the present paper concern lower bounds. First an abstract setting of algorithms and n-th minimal errors is introduced, which extends respective approaches for deterministic problems. Then we prove lower bounds for adaptive algorithms both in the deterministic and randomized setting matching the upper bounds derived before (up to logarithmic factors, in general). We present a new technique, which involves exponential inequalities. It is also shown that the bounds obtained in [2] for nonadaptive deterministic algorithms hold true for adaptive deterministic and randomized algorithms, as well.

The structure of the paper is as follows: Section 2 contains notation and some preliminaries, including the needed function classes. In section 3 we recall the multilevel Euler-Maruyama algorithm from [2] and derive error estimates. Section 4 is devoted to lower bounds.

2 Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Let X, Y be Banach spaces. The unit ball of X is denoted by B_X, the dual space by X^*, the σ-algebra of Borel subsets of X by $\mathscr{B}(X)$, and the space of bounded linear operators from Y to X by $\mathscr{L}(Y, X)$. Let $d \in \mathbb{N}$. The space of real-valued continuous functions on a compact set $Q \subset \mathbb{R}^d$ is denoted by $C(Q)$ and is equipped with the supremum norm. Furthermore, if Q is the closure of an open bounded set and $k \in \mathbb{N}$, $C^k(Q)$ denotes the space of all functions which are k-times continuously differentiable in the interior of Q and which together with their derivatives up to order k possess continuous extensions to all of Q. This space is equipped with the norm $\|f\|_{C^k(Q)} = \sup_{|\alpha| \le k,\, s \in Q} |D^\alpha f(s)|$ with $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $|\alpha| = |\alpha_1| + \dots + |\alpha_d|$. If $k = 0$, we put $C^0(Q) = C(Q)$. For $r \in \mathbb{R}$, $r > 0$, $r \notin \mathbb{N}$ put $k = \lfloor r \rfloor$, $\sigma = r - k$, and let $C^r(Q)$ be the space of all $f \in C^k(Q)$ satisfying $\|f\|_{C^r(Q)} < \infty$, where

$$\|f\|_{C^r(Q)} \;:=\; \max\left(\|f\|_{C^k(Q)}, \max_{|\alpha| \le k} \sup_{s_1 \ne s_2 \in Q} |s_1 - s_2|^{-\sigma} \left| D^\alpha f(s_1) - D^\alpha f(s_2) \right| \right),$$

and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. For $1 \le p < \infty$ and $(M, \mathscr{M}, \mu)$ an arbitrary measure space, $L_p(M, \mathscr{M}, \mu, X)$, or shortly $L_p(M, X)$, is the space of Bochner p-integrable functions, equipped with the usual norm.

Throughout the paper the same symbol $c, c_1, c_2, \dots$ may denote different constants, even in a sequence of relations. Moreover, for nonnegative reals $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ we write $a_n \preceq b_n$ if there are constants $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $a_n \le c b_n$. Furthermore, $a_n \asymp b_n$ means that $a_n \preceq b_n$ and $b_n \preceq a_n$. Finally, $a_n \preceq_{\log} b_n$ iff there are constants $c > 0$, $n_0 \in \mathbb{N}$, and $\theta \in \mathbb{R}$ such that for all $n \ge n_0$ $a_n \le c b_n (\log(n+1))^\theta$.

Let $Q = [0,1]^d$, $k \in \mathbb{N}$, $m \in \mathbb{N}$, and let $\Gamma_m^{k,d} = \left\{ \frac{i}{km} : 0 \le i \le km \right\}^d$. Let $P_m^{k,d} \in \mathscr{L}(C(Q))$ be tensor product composite Lagrange interpolation of degree k with respect to the partition of Q given by $\Gamma_m^{1,d}$. Let $r \in \mathbb{R}$, $r > 0$, and set

$k = \lceil r \rceil$. It is well-known that there are constants $c_0, c_1 > 0$ such that for all $m \in \mathbb{N}$

$$\|P_m^{k,d}\|_{\mathscr{L}(C(Q))} \le c_0, \quad \|J - P_m^{k,d} J\|_{\mathscr{L}(C^r(Q),C(Q))} \le c_1 m^{-r}, \tag{1}$$

where $J : C^r(Q) \to C(Q)$ is the embedding.

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, $(\Sigma_t)_{0\le t\le 1}$, $\Sigma_t \subseteq \Sigma$ a filtration, let $(W(t))_{0\le t\le 1}$, $W(t) = W(t,\omega)$ $(\omega \in \Omega)$ be a Wiener process on $(\Omega, \Sigma, \mathbb{P})$ adapted to (Σ_t) and such that for $0 \le t_1 \le t_2 \le 1$ the increments $W(t_2) - W(t_1)$ are independent of Σ_{t_1}. We assume that all trajectories of the Wiener process are continuous.

Next we introduce the class of random functions which we will study here. Let $r \in \mathbb{R}$, $r > 0$, $d \in \mathbb{N}$, $Q = [0,1]^d$, $2 \le q < \infty$ and let $\mathscr{F}_q^r = \mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa)$ denote the set of all functions $f : Q \times [0,1] \times \Omega \to \mathbb{R}$ such that for each $s \in Q$, $f(s,t,\omega)$ is progressively measurable, in other words, for each $\tau \in [0,1]$ the restriction $f(s,\cdot,\cdot)|_{[0,\tau]\times\Omega}$ is $\mathscr{B}([0,\tau]) \times \Sigma_\tau$ measurable,

$$\begin{aligned} f(\,\cdot\,,\,\cdot\,,\omega) &\in C^r(Q \times [0,1]) \qquad (\omega \in \Omega), &(2)\\ \left(\mathbb{E}\,\|f(\,\cdot\,,\,\cdot\,,\omega)\|^q_{C^r(Q\times[0,1])}\right)^{1/q} &\le \kappa. &(3)\end{aligned}$$

We need to recall the definition of related classes in [2]. Let $r_1 \in \mathbb{R}$, $r_1 > 0$, $0 \le \varrho \le 1$ and let $F^{r_1,\varrho} = F^{r_1,\varrho}(Q \times [0,1] \times \Omega; \kappa)$ denote the set of all functions $f : Q \times [0,1] \times \Omega \to \mathbb{R}$ such that for each $s \in Q$, $f(s,t,\omega)$ is progressively measurable and

$$\begin{aligned} f(\,\cdot\,,t,\omega) &\in C^{r_1}(Q) \quad ((t,\omega) \in [0,1] \times \Omega), &(4)\\ \left(\mathbb{E}\,\|f(\,\cdot\,,0,\omega)\|^2_{C^{r_1}(Q)}\right)^{1/2} &\le \kappa, &(5)\\ \left(\mathbb{E}\,\|f(\cdot,t_1,\omega) - f(\cdot,t_2,\omega)\|^2_{C^{r_1}(Q)}\right)^{1/2} &\le \kappa|t_1 - t_2|^\varrho \quad (t_1,t_2 \in [0,1]). &(6)\end{aligned}$$

Let $F^{r_1,\varrho}(Q \times [0,1] \times \Omega) = \cup_{\kappa>0} F^{r_1,\varrho}(Q \times [0,1] \times \Omega; \kappa)$ be the respective linear space. Moreover, for $2 < q < \infty$ let $F_q^{r_1,\varrho} = F_q^{r_1,\varrho}(Q \times [0,1] \times \Omega; \kappa)$ be the subset of those $f \in F^{r_1,\varrho}(Q \times [0,1] \times \Omega; \kappa)$ which fulfill

$$\left(\mathbb{E}\max_{t\in M}\|f(\,\cdot\,,t,\omega)\|^q_{C^{r_1}(Q)}\right)^{1/q} \le \kappa \quad (M \subset [0,1], |M| < \infty). \tag{7}$$

Let us consider the relation between the two types of function classes.

Lemma 1. *Let $r, r_1 > 0$, $0 \le \varrho \le 1$, $r \ge r_1 + \varrho$, $2 < q < \infty$. Then there are constants $c_1, c_2 > 0$ such that*

$$\begin{aligned} \mathscr{F}_2^r(Q \times [0,1] \times \Omega; \kappa) &\subseteq F^{r_1,\varrho}(Q \times [0,1] \times \Omega; c_1\kappa) &(8)\\ \mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa) &\subseteq F_q^{r_1,\varrho}(Q \times [0,1] \times \Omega; c_2\kappa). &(9)\end{aligned}$$

Proof. Let $2 \le q < \infty$ and $f \in \mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa)$. Clearly, (4) follows from (2), while (3) implies (5) and (7). It remains to show that (6) holds. We can assume $r = r_1 + \varrho$. Let

$$r = k + \sigma, \quad r_1 = k_1 + \sigma_1 \quad (k, k_1 \in \mathbb{N}_0,\ 0 \le \sigma, \sigma_1 < 1).$$

Fix $\omega \in \Omega$ and set

$$\kappa(\omega) = \|f(\,\cdot\,,\,\cdot\,,\omega)\|_{C^r(Q\times[0,1])}\,.$$

Let $s_1, s_2 \in Q$, $s_1 \ne s_2$, $t_1, t_2 \in [0,1]$, $t_1 \ne t_2$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le k_1$, and put

$$\begin{aligned} \Delta_1(t_1,t_2,\omega) &:= \|f(\,\cdot\,,t_1,\omega) - f(\,\cdot\,,t_2,\omega)\|_{C^{k_1}(Q)} \\ \Delta_2^\alpha(s_1,s_2,t_1,t_2,\omega) &:= |D_s^\alpha f(s_1,t_1,\omega) - D_s^\alpha f(s_1,t_2,\omega) \\ &\qquad -D_s^\alpha f(s_2,t_1,\omega) + D_s^\alpha f(s_2,t_2,\omega)|. \end{aligned}$$

First we assume $\sigma_1 + \varrho = \sigma$, thus $k_1 = k$. Then, taking into account $|t_1 - t_2| \le 1$,

$$\Delta_1(t_1,t_2,\omega) \le \kappa(\omega)|t_1-t_2|^\sigma \le \kappa(\omega)|t_1-t_2|^\varrho. \tag{10}$$

If $|t_1 - t_2| \ge |s_1 - s_2|$, then

$$\Delta_2^\alpha(s_1,s_2,t_1,t_2,\omega) \le 2\kappa(\omega)|s_1-s_2|^\sigma \le 2\kappa(\omega)|s_1-s_2|^{\sigma_1}|t_1-t_2|^\varrho. \tag{11}$$

Similarly, if $|t_1 - t_2| < |s_1 - s_2|$, then

$$\Delta_2^\alpha(s_1,s_2,t_1,t_2,\omega) \le 2\kappa(\omega)|t_1-t_2|^\sigma \le 2\kappa(\omega)|s_1-s_2|^{\sigma_1}|t_1-t_2|^\varrho. \tag{12}$$

Now we assume $\sigma_1 + \varrho = 1 + \sigma$, hence $k_1 = k - 1$. Here we have

$$\Delta_1(t_1,t_2,\omega) \le \kappa(\omega)|t_1-t_2| \le \kappa(\omega)|t_1-t_2|^\varrho. \tag{13}$$

Let e_i be the i-th unit vector in $\mathbb{R}^d$. If $|t_1 - t_2| \ge |s_1 - s_2|$, then

$$\begin{aligned} &\Delta_2^\alpha(s_1,s_2,t_1,t_2,\omega) \\ &= \Bigg\| \int_0^1 \sum_{i=1}^d \Big(D_s^{\alpha+e_i} f(s_2 + \theta(s_1-s_2), t_1, \omega) \\ &\qquad -D_s^{\alpha+e_i} f(s_2 + \theta(s_1-s_2), t_2, \omega)\Big)(s_{1,i} - s_{2,i})d\theta \Bigg\|_{C^{k_1}(Q)} \\ &\le \sqrt{d}\kappa(\omega)|s_1-s_2||t_1-t_2|^\sigma \le \sqrt{d}\kappa(\omega)|s_1-s_2|^{\sigma_1}|t_1-t_2|^\varrho. \end{aligned} \tag{14}$$

Similarly, if $|t_1 - t_2| < |s_1 - s_2|$, then

$$\begin{aligned} &\Delta_2^\alpha(s_1,s_2,t_1,t_2,\omega) \\ &= \Bigg\| \int_0^1 \Big(D_{s,t}^{\alpha,1} f(s_1, t_2 + \theta(t_1-t_2), \omega) \\ &\qquad -D_{s,t}^{\alpha,1} f(s_2, t_2 + \theta(t_1-t_2), \omega)\Big)(t_1 - t_2)d\theta \Bigg\|_{C^{k_1}(Q)} \\ &\le \kappa(\omega)|t_1-t_2||s_1-s_2|^\sigma \le \kappa(\omega)|s_1-s_2|^{\sigma_1}|t_1-t_2|^\varrho. \end{aligned} \tag{15}$$

It follows from (10–15) that

$$\begin{aligned}
&\left(\mathbb{E}\,\|f(\,\cdot\,,t_1,\omega)-f(\,\cdot\,,t_2,\omega)\|^2_{C^{r_1}(Q)}\right)^{1/2}\\
&= \left(\mathbb{E}\max\left(\Delta_1(t_1,t_2,\omega),\max_{|\alpha|\le k_1}\sup_{s_1\ne s_2\in Q}\frac{\Delta_2^\alpha(s_1,s_2,t_1,t_2,\omega)}{|s_1-s_2|^{\sigma_1}}\right)^2\right)^{1/2}\\
&\le \max(\sqrt{d},2)\left(\mathbb{E}\,\kappa(\omega)^2\right)^{1/2}|t_1-t_2|^\varrho \le \max(\sqrt{d},2)\kappa|t_1-t_2|^\varrho,
\end{aligned}$$

which shows (6). □

Now we consider parametric indefinite stochastic integration

$$\int_0^t f(s,\tau)dW(\tau) \quad (s\in Q, t\in[0,1]).$$

This is a stochastic process indexed by $Q\times[0,1]$. It was shown in [2] (for $r\in\mathbb{N}$, but the argument is the same for real $r>0$) that we can find a continuous version in the sense that there is a mapping

$$\hat{\mathscr{S}}: F^{r,0}(Q\times[0,1]\times\Omega)\to L_2(\Omega, C(Q\times[0,1]))$$

such that for $s\in Q$, $t\in[0,1]$

$$(\hat{\mathscr{S}}(f))(s,t) \;=\; \int_0^t f(s,\tau)dW(\tau). \tag{16}$$

It follows from the linearity of the stochastic integral (and a standard density/continuity argument) that the operator $\hat{\mathscr{S}}$ is linear. For our purposes we need a mapping

$$\mathscr{S}: F^{r,0}(Q\times[0,1]\times\Omega)\times\Omega\to C(Q\times[0,1]) \tag{17}$$

such that $\mathscr{S}(f,\,\cdot\,)=\hat{\mathscr{S}}(f)$, with equality meant in $L_2(\Omega,C(Q\times[0,1]))$, and $\mathscr{S}$ is linear in f. Let $(f_i)_{i\in I}$, I a suitable index set, be a Hamel basis (that is, a basis in the sense of linear spaces) of $F^{r,0}(Q\times[0,1]\times\Omega)$. For each $i\in I$ let $g_i=g_i(\omega)$ be a representative of the equivalence class $\hat{\mathscr{S}}(f_i)\in L_2(\Omega,C(Q\times[0,1]))$. Then we set $\mathscr{S}(f_i,\omega)=g_i(\omega)$ for $i\in I$ and $\omega\in\Omega$ and extend the so-defined mapping by linearity to all of $F^{r,0}(Q\times[0,1]\times\Omega)$. It follows from the linearity of $\hat{\mathscr{S}}$ that $\mathscr{S}$ is as required.

For parametric definite stochastic integration

$$\int_0^1 f(s,\tau)dW(\tau) \quad (s\in Q)$$

we define

$$\mathscr{S}_1: F^{r,0}(Q\times[0,1]\times\Omega)\times\Omega\to C(Q)$$

by setting

$$(\mathscr{S}_1(f,\omega))(s) \quad = \quad (\mathscr{S}(f,\omega))(s,1) \quad (s \in Q, \omega \in \Omega). \tag{18}$$

It follows that

$$\mathscr{S}_1(f,\,\cdot\,) \quad \in \quad L_2(\Omega, C(Q)) \tag{19}$$

$$(\mathscr{S}_1(f,\,\cdot\,))(s) \quad = \quad \int_0^1 f(s,\tau)dW(\tau) \quad (s \in Q), \tag{20}$$

with equality (20) meant in $L_2(\Omega)$. Due to Lemma 1, the operators $\mathscr{S}$ and $\mathscr{S}_1$ are also defined on the respective sets $\mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa)$.

3 An Algorithm for Parametric Stochastic Integrals

First we recall the algorithm from [2]. Let $n \in \mathbb{N}$, $t_k = k/n$ $(k = 0,\ldots,n)$, and define $A_n(f,\omega) \in C([0,1])$ for any function $f : [0,1 \times \Omega \to \mathbb{R}$ and $\omega \in \Omega$ by

$$A_n(f,\omega) \quad = \quad P_n^{1,1}\left(\sum_{j=0}^{k-1} f(t_j,\omega)(W(t_{j+1},\omega) - W(t_j,\omega))\right)_{k=0}^{n}.$$

This is the piecewise linear interpolation of the Euler-Maruyama scheme. Furthermore, we set

$$A_{n,1}(f,\omega) \quad = \quad (A_n(f,\omega))(1) = \sum_{j=0}^{n-1} f(t_j,\omega)(W(t_{j+1},\omega) - W(t_j,\omega)).$$

Next we pass to the multilevel scheme of [2]. Put $k = \lceil r \rceil$, fix $l_1 \in \mathbb{N}_0$, $n_0,\ldots,n_{l_1} \in \mathbb{N}$, let $f : Q \times [0,1] \times \Omega \to \mathbb{R}$ be any function and $\omega \in \Omega$. For the indefinite problem we set

$$\mathscr{A}(f,\omega) = \sum_{l=0}^{l_1} \left(P_{2^l}^{k,d} - P_{2^{l-1}}^{k,d}\right) \left(A_{n_l}(f_s,\omega)\right)_{s \in \Gamma_{2^l}^{k,d}},$$

where f_s is given by $f_s(t,\omega) := f(s,t,\omega)$ $(t \in [0,1],\ \omega \in \Omega)$ and $P_{2^{-1}} := 0$. In the definite case we put

$$\mathscr{A}_1(f,\omega) = \sum_{l=0}^{l_1} \left(P_{2^l}^{k,d} - P_{2^{l-1}}^{k,d}\right) \left(A_{n_l,1}(f_s,\omega)\right)_{s \in \Gamma_{2^l}^{k,d}}.$$

Let $\operatorname{card}(\mathscr{A})$ denote the number of evaluations of f and W used in algorithm $\mathscr{A}$ (see Section 4 for a general definition). We have

$$\operatorname{card}(\mathscr{A}) = \operatorname{card}(\mathscr{A}_1) \leq c \sum_{l=0}^{l_1} n_l 2^{dl}.$$

Now we use the analysis of this algorithm given in [2]. The following is Theorem 5.3 of [2], which was shown there for $r_1 \in \mathbb{N}$. It is easily seen that it also holds for real $r_1 > 0$. Indeed, Lemma 5.1 and, using (1) above, Proposition 5.2 of [2] are readily extended to non-integer $r_1 > 0$. The proof of Theorem 5.3 relies only on Proposition 5.2 and does not use the assumption of r_1 being integer.

Theorem 1. *Let $r_1 \in \mathbb{R}$, $r_1 > 0$, $d \in \mathbb{N}$, $0 \le \varrho \le 1$, $2 < q < \infty$. There are constants $c_{1-4} > 0$ such that the following hold. For each $n \in \mathbb{N}$ with $n \ge 2$ there is a choice of $l_1 \in \mathbb{N}_0$ and $n_0, \dots, n_{l_1} \in \mathbb{N}_0$ such that* $\operatorname{card}(\mathscr{A}_1) \le c_1 n$ *and*

$$\sup_{f \in F^{r_1,\varrho}(Q\times[0,1]\times\Omega;\kappa)} \left(\mathbb{E}\,\|\mathscr{S}_1(f,\omega) - \mathscr{A}_1(f,\omega)\|^2_{C(Q)}\right)^{1/2} \le c_2 \begin{cases} n^{-r_1/d}(\log n)^{1/2} & \text{if } r_1/d < \varrho, \\ n^{-r_1/d}(\log n)^{r_1/d+3/2} & \text{if } r_1/d = \varrho, \\ n^{-\varrho} & \text{if } r_1/d > \varrho. \end{cases} \tag{21}$$

Similarly, for each $n \in \mathbb{N}$ with $n \ge 2$ there are $l_1 \in \mathbb{N}_0$ and $n_0, \dots, n_{l_1} \in \mathbb{N}_0$ such that $\operatorname{card}(\mathscr{A}) \le c_3 n$ *and*

$$\sup_{f \in F_q^{r_1,\varrho}(Q\times[0,1]\times\Omega;\kappa)} \left(\mathbb{E}\,\|\mathscr{S}(f,\omega) - \mathscr{A}(f,\omega)\|^2_{C(Q\times[0,1])}\right)^{1/2} \le c_4 \begin{cases} n^{-r_1/d}(\log n)^{1/2} & \text{if } r_1/d < \min(\varrho,1/2) \\ n^{-r_1/d}(\log n)^{r_1/d+3/2} & \text{if } r_1/d = \min(\varrho,1/2), \\ n^{-1/2}(\log n)^{1/2} & \text{if } r_1/d > \min(\varrho,1/2),\ \varrho \ge 1/2, \\ n^{-\varrho} & \text{if } r_1/d > \min(\varrho,1/2),\ \varrho < 1/2. \end{cases} \tag{22}$$

On the basis of the considerations above we can now derive error estimates for algorithms $\mathscr{A}_1$ and $\mathscr{A}$ on the classes $\mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa)$.

Theorem 2. *Let $r \in \mathbb{R}$, $r > 0$, $d \in \mathbb{N}$, $2 < q < \infty$, $\kappa > 0$. Then there are constants $c_{1-4} > 0$ such that for each $n \in \mathbb{N}$ with $n \ge 2$ there is a choice of $l_1 \in \mathbb{N}_0$ and $n_0, \dots, n_{l_1} \in \mathbb{N}_0$ such that* $\operatorname{card}(\mathscr{A}_1) \le c_1 n$ *and*

$$\sup_{f \in \mathscr{F}_2^r(Q\times[0,1]\times\Omega;\kappa)} \left(\mathbb{E}\,\|\mathscr{S}_1(f,\omega) - \mathscr{A}_1(f,\omega)\|^2_{C(Q)}\right)^{1/2} \le c_2 \begin{cases} n^{-\frac{r}{d+1}}(\log n)^{\frac{r}{d+1}+\frac{3}{2}} & \text{if } \frac{r}{d+1} \le 1, \\ n^{-1} & \text{if } \frac{r}{d+1} > 1. \end{cases} \tag{23}$$

Moreover, for each $n \in \mathbb{N}$ with $n \ge 2$ there are $l_1 \in \mathbb{N}_0$ and $n_0, \dots, n_{l_1} \in \mathbb{N}_0$ such that $\operatorname{card}(\mathscr{A}) \le c_3 n$ *and*

$$\sup_{f \in \mathscr{F}_q^r(Q\times[0,1]\times\Omega;\kappa)} \left(\mathbb{E}\,\|\mathscr{S}(f,\omega) - \mathscr{A}(f,\omega)\|^2_{C(Q\times[0,1])}\right)^{1/2} \le c_4 \begin{cases} n^{-\frac{r}{d+1}}(\log n)^{\frac{r}{d+1}+\frac{3}{2}} & \text{if } \frac{r}{d+1} \le 1/2, \\ n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} & \text{if } \frac{r}{d+1} > \frac{1}{2}. \end{cases} \tag{24}$$

Proof. We define

$$r_1 = \frac{dr}{d+1}, \quad \varrho = \min\left(\frac{r}{d+1}, 1\right), \tag{25}$$

hence $r \geq r_1 + \varrho$ and the conclusions (8–9) of Lemma 1 hold. Denote the left-hand side of (23) and (24) by E_1 and E, respectively.

First we derive (23). If $\frac{r}{d+1} \leq 1$, then $\frac{r_1}{d} = \varrho$, thus (8) together with the second relation of (21) yields

$$E_1 \leq cn^{-\frac{r_1}{d}} (\log n)^{\frac{r_1}{d}+\frac{3}{2}} = cn^{-\frac{r}{d+1}} (\log n)^{\frac{r}{d+1}+\frac{3}{2}},$$

which is the first relation of (23). If $\frac{r}{d+1} > 1$, we have by (25) $\frac{r_1}{d} > \varrho = 1$, so the third relation of (21) gives $E_1 \leq cn^{-1}$ and thus the second part of (23).

Next we prove (24). If $\frac{r}{d+1} \leq \frac{1}{2}$, we conclude from (25) $\frac{r_1}{d} = \varrho = \min(\varrho, 1/2)$, so (9) together with the second relation of (22) implies

$$E \leq cn^{-\frac{r_1}{d}} (\log n)^{\frac{r_1}{d}+\frac{3}{2}} = cn^{-\frac{r}{d+1}} (\log n)^{\frac{r}{d+1}+\frac{3}{2}},$$

thus the first relation of (24). Finally, if $\frac{r}{d+1} > \frac{1}{2}$, then by (25), $\varrho > 1/2$, hence $\frac{r_1}{d} > \min(\varrho, 1/2)$, and the third relation of (22) implies

$$E \leq cn^{-\frac{1}{2}} (\log n)^{\frac{1}{2}},$$

showing the second relation of (24) and completing the proof. □

4 Lower Bounds and Complexity

In this section we extend the approach of [6], [7] to stochastic problems. An abstract stochastic numerical problem is described by a tuple

$$\mathscr{P} = (F, (\Omega, \Sigma, \mathbb{P}), G, S, K, \Lambda). \tag{26}$$

The set F is an arbitrary non-empty set, $(\Omega, \Sigma, \mathbb{P})$ a probability space, G is a Banach space and $S : F \times \Omega \to G$ an arbitrary mapping, the solution operator, which maps the input $(f, \omega) \in F$ to the exact solution $S(f, \omega)$. We assume that for each $f \in F$ the mapping $\omega \to S(f, \omega)$ is Σ-to-Borel-measurable and $\mathbb{P}$-almost surely separably valued, the latter meaning that for each $f \in F$ there is a separable subspace G_f of G such that

$$\mathbb{P}\{\omega : \, S(f, \omega) \in G_f\} = 1.$$

Furthermore, K is a nonempty set and Λ a set of mappings from $F \times \Omega$ to K, the set of information functionals.

A deterministic algorithm for $\mathscr{P}$ is a tuple $A = ((L_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty)$ such that $L_1 \in \Lambda$, $\tau_0 \in \{0,1\}$, $\varphi_0 \in G$, and for $i \in \mathbb{N}$

$$L_{i+1} : K^i \to \Lambda, \quad \tau_i : K^i \to \{0,1\}, \quad \varphi_i : K^i \to G$$

are arbitrary mappings. Given $(f,\omega) \in F \times \Omega$, we associate with it a sequence $(a_i)_{i=1}^\infty$ defined as follows:

$$a_1 = L_1(f,\omega) \tag{27}$$
$$a_i = \big(L_i(a_1,\dots,a_{i-1})\big)(f,\omega) \quad (i \ge 2). \tag{28}$$

Define $\operatorname{card}(A,f,\omega)$, the cardinality of A at input (f,ω), to be 0 if $\tau_0 = 1$. If $\tau_0 = 0$, let $\operatorname{card}(A,f,\omega)$ be the first integer $n \in \mathbb{N}$ with $\tau_n(a_1,\dots,a_n) = 1$, if there is such an n. If no such $n \in \mathbb{N}$ exists, set $\operatorname{card}(A,f,\omega) = \infty$. We define the output $A(f,\omega)$ of algorithm A at input (f,ω) as

$$A(f,\omega) = \begin{cases} \varphi_0 & \text{if} \quad \operatorname{card}(A,f,\omega) = 0 \\ \varphi_n(a_1,\dots,a_n) & \text{if} \quad \operatorname{card}(A,f,\omega) = n < \infty \\ \varphi_0 & \text{if} \quad \operatorname{card}(A,f,\omega) = \infty. \end{cases}$$

Given $n \in \mathbb{N}_0$, we define $\mathscr{A}_n^{\mathrm{det}}(\mathscr{P})$ as the set of those deterministic algorithms A for $\mathscr{P}$ with the following properties: For each $f \in F$ the mapping $\omega \to \operatorname{card}(A,f,\omega)$ is Σ-measurable, $\mathbb{E}\operatorname{card}(A,f,\omega) \le n$, and the mapping $\omega \to A(f,\omega) \in G$ is Σ-to-Borel-measurable and $\mathbb{P}$-almost surely separably valued. The cardinality of $A \in \mathscr{A}_n^{\mathrm{det}}(\mathscr{P})$ is defined as

$$\operatorname{card}(A) = \sup_{f\in F} \mathbb{E}\operatorname{card}(A,f,\omega),$$

the error of A in approximating S as

$$e(S,A,F\times\Omega,G) = \sup_{f\in F} \mathbb{E}\,\|S(f,\omega) - A(f,\omega)\|_G$$

and the deterministic n-th minimal error of S is defined for $n \in \mathbb{N}_0$ as

$$e_n^{\mathrm{det}}(S,F\times\Omega,G) = \inf_{A\in\mathscr{A}_n^{\mathrm{det}}(\mathscr{P})} e(S,A,F\times\Omega,G).$$

It follows that no deterministic algorithm that uses (on the average with respect to $\mathbb{P}$) at most n information functionals can have a smaller error than $e_n^{\mathrm{det}}(S,F\times\Omega,G)$.

A randomized algorithm for $\mathscr{P}$ is a tuple $A = ((\Omega_1,\Sigma_1,\mathbb{P}_1),\ (A_{\omega_1})_{\omega_1\in\Omega_1})$, where $(\Omega_1,\Sigma_1,\mathbb{P}_1)$ is another probability space and for each $\omega_1 \in \Omega_1$, A_{ω_1} is a deterministic algorithm for $\mathscr{P}$. Let $(\Omega_1\times\Omega, \Sigma_1\times\Sigma, \mathbb{P}_1\times\mathbb{P})$ be the product probability space. For $n \in \mathbb{N}_0$ we define $\mathscr{A}_n^{\mathrm{ran}}(\mathscr{P})$ as the class of those randomized algorithms A for $\mathscr{P}$ which possess the following properties: For each $f \in F$ the mapping $(\omega_1, \omega) \to \operatorname{card}(A_{\omega_1}, f, \omega)$ is $\Sigma_1 \times \Sigma$-measurable,

$$\mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}} \operatorname{card}(A_{\omega_1}, f, \omega) \leq n,$$

and the mapping $(\omega_1, \omega) \to A_{\omega_1}(f, \omega)$ is $\Sigma_1 \times \Sigma$-to-Borel-measurable and $\mathbb{P}_1 \times \mathbb{P}$-almost surely separably valued. We define the cardinality of $A \in \mathscr{A}_n^{\mathrm{ran}}(\mathscr{P})$ as

$$\operatorname{card}(A) = \sup_{f \in F} \mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}} \operatorname{card}(A_{\omega_1}, f, \omega),$$

the error as

$$e(S, A, F \times \Omega, G) = \sup_{f \in F} \mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}} \| S(f, \omega) - A_{\omega_1}(f, \omega) \|_G$$

and the randomized n-th minimal error of S as

$$e_n^{\mathrm{ran}}(S, F \times \Omega, G) = \inf_{A \in \mathscr{A}_n^{\mathrm{ran}}(\mathscr{P})} e(S, A, F \times \Omega, G).$$

Similarly to the above, this means that no randomized algorithm that uses (on the average with respect to $\mathbb{P}_1 \times \mathbb{P}$) at most n information functionals can have a smaller error than $e_n^{\mathrm{ran}}(S, F \times \Omega, G)$. Deterministic algorithms can be viewed as a special case of randomized ones, namely by considering trivial one-point probability spaces $\Omega_1 = \{\omega_1\}$. Hence,

$$e_n^{\mathrm{ran}}(S, F \times \Omega, G) \leq e_n^{\mathrm{det}}(S, F \times \Omega, G). \tag{29}$$

Now we study the complexity of definite and indefinite stochastic integration. Let $r, r_1 > 0$, $0 \leq \varrho \leq 1$, $2 < q < \infty$. We set $K = \mathbb{R}$ and

$$\Lambda = \Lambda_1 \cup \Lambda_2, \quad \Lambda_1 = \{\delta_{st} : s \in Q, t \in [0,1]\}, \quad \Lambda_2 = \{\delta_t : t \in [0,1]\}, \tag{30}$$

where $\delta_{st}(f, \omega) = f(s, t, \omega)$ and $\delta_t(f, \omega) = W(t, \omega)$ $(f \in F, \omega \in \Omega)$. For definite integration we choose $F = \mathscr{F}_2^r(Q \times [0,1] \times \Omega; \kappa)$ or $F = F^{r_1, \varrho}(Q \times [0,1] \times \Omega; \kappa)$, $G = C(Q)$, $S = \mathscr{S}_1$. For the indefinite problem we set $F = \mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa)$ or $F = F_q^{r_1, \varrho}(Q \times [0,1] \times \Omega; \kappa)$, $G = C(Q \times [0,1])$, $S = \mathscr{S}$.

Theorem 3. *Let $r, r_1 \in \mathbb{R}$, $r, r_1 > 0$, $0 \leq \varrho \leq 1$, $d \in \mathbb{N}$, $\kappa > 0$, and $2 < q < \infty$. Then*

$$e_n^{\mathrm{ran}}(\mathscr{S}_1, \mathscr{F}_2^r \times \Omega, C(Q)) \succeq \max\left(n^{-\frac{r}{d+1}}, n^{-1}\right) \tag{31}$$

$$e_n^{\mathrm{ran}}(\mathscr{S}_1, F^{r_1, \varrho} \times \Omega, C(Q)) \succeq \max\left(n^{-\frac{r_1}{d}}, n^{-\varrho}\right). \tag{32}$$

$$e_n^{\mathrm{ran}}(\mathscr{S}, \mathscr{F}_q^r \times \Omega, C(Q \times [0,1]))) \succeq \max\left(n^{-\frac{r}{d+1}}, n^{-\frac{1}{2}} (\log n)^{\frac{1}{2}}\right) \tag{33}$$

$$e_n^{\mathrm{ran}}(\mathscr{S}, F_q^{r_1, \varrho} \times \Omega, C(Q \times [0,1])) \succeq \max\left(n^{-\frac{r_1}{d}}, n^{-\frac{1}{2}} (\log n)^{\frac{1}{2}}, n^{-\varrho}\right). \tag{34}$$

Theorems 1 and 2 show that, up to logarithmic factors, these bounds match the upper bounds.

Corollary 1. *Relations (31–34) also hold with* $\succeq$ *replaced by* $\preceq_{\log}$.

Moreover, by (29) and since the algorithms in Theorems 1 and 2 are deterministic, the conclusions of Theorem 3 and Corollary 1 hold for $e_n^{\rm det}$ in place of $e_n^{\rm ran}$, as well.

To prove Theorem 3 we need a number of auxiliary results. For this we return to the general setting (26). The first observation concerns the case that F consists of a single element, in other words, S is essentially independent of F and $\mathscr{P}$ is a pure average case problem. Then the above inequality (29) has a certain converse. This is a version of the well-known principle that, in general, for pure average case problems randomized algorithms do not bring essential gains.

Lemma 2. *If* $F = F_0 = \{f_0\}$, *then*

$$e_n^{\rm ran}(S, F_0 \times \Omega, G) \ge \frac{1}{2} e_{2n}^{\rm det}(S, F_0 \times \Omega, G). \tag{35}$$

Proof. Let $\delta > 0$ and $A \in \mathscr{A}_n^{\rm ran}(\mathscr{P})$ with

$$e(S, A, F_0 \times \Omega, G) \le e_n^{\rm ran}(S, F_0 \times \Omega, G) + \delta.$$

This means

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}} \|S(f_0, \omega) - A_{\omega_1}(f_0, \omega)\|_G &\le e_n^{\rm ran}(S, F_0 \times \Omega, G) + \delta, \\ \mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}} \operatorname{card}(A_{\omega_1}, f_0, \omega) &\le n. \end{aligned}$$

Consequently, setting

$$\begin{aligned} \Omega_{1,1} &= \{\omega_1 : \mathbb{E}_{\mathbb{P}} \|S(f_0, \omega) - A_{\omega_1}(f_0, \omega)\|_G \le 2e_n^{\rm ran}(S, F_0 \times \Omega, G) + 2\delta\}, \\ \Omega_{1,2} &= \{\omega_1 : \mathbb{E}_{\mathbb{P}} \operatorname{card}(A_{\omega_1}, f_0, \omega) \le 2n\}, \end{aligned}$$

we conclude that $\mathbb{P}_1(\Omega_{1,1}) > 1/2$ and $\mathbb{P}_1(\Omega_{1,2}) > 1/2$. It follows that for $\omega_1 \in \Omega_{1,1} \cap \Omega_{1,2} \neq \emptyset$ we have $A_{\omega_1} \in \mathscr{A}_{2n}^{\rm det}(\mathscr{P})$ and

$$e(S, A_{\omega_1}, F_0 \times \Omega, G) \le 2e_n^{\rm ran}(S, F_0 \times \Omega, G) + 2\delta,$$

which implies (35). □

Next we explore the connection between the original stochastic problem and the deterministic problem we obtain by fixing the random input. For this purpose, we assume that we are given a decomposition of the set Λ

$$\Lambda = \Lambda_F \cup \Lambda_\Omega, \quad \Lambda_F \neq \emptyset, \quad \Lambda_F \cap \Lambda_\Omega = \emptyset$$

such that for all $\lambda \in \Lambda_\Omega$ we have $\lambda(f,\omega) = \lambda(g,\omega)$ $(f,g \in F, \omega \in \Omega)$, that is, all $\lambda \in \Lambda_\Omega$ depend only on $\omega \in \Omega$ (the $\lambda \in \Lambda_F$ may depend on both f and ω). For $\lambda \in \Lambda_\Omega$ we use both the notation $\lambda(f,\omega)$ as well as $\lambda(\omega)$. Note that there is always the trivial splitting $\Lambda_F = \Lambda$, $\Lambda_\Omega = \emptyset$. An example of a nontrivial splitting is (30) above. Fix $\omega \in \Omega$. We define the restricted problem $\mathscr{P}_\omega = (F, G, S_\omega, K, \Lambda_{F,\omega})$ by setting

$$S_\omega : F \to G, \quad S_\omega(f) = S(f,\omega), \quad \Lambda_{F,\omega} = \{\lambda(\,\cdot\,,\omega) : \ \lambda \in \Lambda_F\}.$$

To a given a deterministic algorithm A for $\mathscr{P}$ and $\omega \in \Omega$ we want to associate a restricted algorithm A_ω for the respective problem $\mathscr{P}_\omega$ in a rigorous way.

Lemma 3. *Let A be a deterministic algorithm for $\mathscr{P}$ and let $\omega \in \Omega$. Then there is a deterministic algorithm A_ω for $\mathscr{P}_\omega$ such that for all $f \in F$*

$$\operatorname{card}(A_\omega, f) = \operatorname{card}(A, f, \omega), \tag{36}$$

$$A_\omega(f) = A(f,\omega). \tag{37}$$

Proof. Let $\mu_0 \in \Lambda_F$ be any element, let $A = ((L_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty)$, and fix $\omega \in \Omega$. We define $A_\omega = ((L_{i,\omega})_{i=1}^\infty, (\tau_{i,\omega})_{i=0}^\infty, (\varphi_{i,\omega})_{i=0}^\infty)$ and a sequence $(\xi_i)_{i=1}^\infty$ of functions $\xi_i : K^i \to K^i$ by induction. Put

$$\tau_{0,\omega} = \tau_0, \quad \varphi_{0,\omega} = \varphi_0, \tag{38}$$

$$L_{1,\omega} = \begin{cases} L_1(\,\cdot\,,\omega) & \text{if} \quad L_1 \in \Lambda_F \\ \mu_0(\,\cdot\,,\omega) & \text{if} \quad L_1 \in \Lambda_\Omega \end{cases} \tag{39}$$

and define for $z_1 \in K$

$$\xi_1(z_1) = \begin{cases} z_1 & \text{if} \quad L_1 \in \Lambda_F \\ L_1(\omega) & \text{if} \quad L_1 \in \Lambda_\Omega. \end{cases} \tag{40}$$

Now let $i \ge 1$ and assume that $(L_{j,\omega})_{j\le i}$, $(\tau_{j,\omega})_{j<i}$, $(\varphi_{j,\omega})_{j<i}$, and $(\xi_j)_{j\le i}$ have been defined. Let $z_1, \dots, z_i, z_{i+1} \in K$ and set

$$\lambda_{i+1} = L_{i+1}(\xi_i(z_1,\dots,z_i)) \tag{41}$$

$$L_{i+1,\omega}(z_1,\dots,z_i) = \begin{cases} \lambda_{i+1}(\,\cdot\,,\omega) & \text{if} \quad \lambda_{i+1} \in \Lambda_F \\ \mu_0(\,\cdot\,,\omega) & \text{if} \quad \lambda_{i+1} \in \Lambda_\Omega \end{cases} \tag{42}$$

$$\tau_{i,\omega}(z_1,\dots,z_i) = \tau_i(\xi_i(z_1,\dots,z_i)) \tag{43}$$

$$\varphi_{i,\omega}(z_1,\dots,z_i) = \varphi_i(\xi_i(z_1,\dots,z_i)) \tag{44}$$

$$\xi_{i+1}(z_1,\dots,z_i,z_{i+1}) = \begin{cases} (\xi_i(z_1,\dots,z_i), z_{i+1}) & \text{if} \quad \lambda_{i+1} \in \Lambda_F \\ (\xi_i(z_1,\dots,z_i), \lambda_{i+1}(\omega)) & \text{if} \quad \lambda_{i+1} \in \Lambda_\Omega. \end{cases} \tag{45}$$

Now let $f \in F$, let $(a_i)_{i=1}^\infty$ be the sequence given by (27) and (28), and define, respectively

$$a_{1,\omega} = L_{1,\omega}(f) \tag{46}$$

$$a_{i,\omega} = \big(L_{i,\omega}(a_{1,\omega},\dots,a_{i-1,\omega})\big)(f) \quad (i \ge 2). \tag{47}$$

We show by induction that for all $i \in \mathbb{N}$.

$$\xi_i(a_{1,\omega}, \dots, a_{i,\omega}) = (a_1, \dots, a_i). \tag{48}$$

For $i = 1$ this follows directly from (27), (39), (40), and (46). Now let $i \in \mathbb{N}$ and assume that (48) holds. Let

$$\lambda_{i+1} = L_{i+1}(\xi_i(a_{1,\omega}, \dots, a_{i,\omega})) = L_{i+1}(a_1, \dots, a_i). \tag{49}$$

First assume $\lambda_{i+1} \in \Lambda_F$. Then by (47), (42), (49), and (28)

$$\begin{aligned} a_{i+1,\omega} &= \big(L_{i+1,\omega}(a_{1,\omega}, \dots, a_{i,\omega})\big)(f) = \lambda_{i+1}(f, \omega) \\ &= \big(L_{i+1}(a_1, \dots, a_i)\big)(f, \omega) = a_{i+1}. \end{aligned}$$

With (45) this gives

$$\xi_{i+1}(a_{1,\omega}, \dots, a_{i,\omega}, a_{i+1,\omega}) = (\xi_i(a_{1,\omega}, \dots, a_{i,\omega}), a_{i+1,\omega}) = (a_1, \dots, a_i, a_{i+1}).$$

In the case $\lambda_{i+1} \in \Lambda_\Omega$ we have, using (49) and (28)

$$\lambda_{i+1}(\omega) = \lambda_{i+1}(f, \omega) = \big(L_{i+1}(a_1, \dots, a_i)\big)(f, \omega) = a_{i+1}.$$

By (45),

$$\xi_{i+1}(a_{1,\omega}, \dots, a_{i,\omega}, a_{i+1,\omega}) = (\xi_i(a_{1,\omega}, \dots, a_{i,\omega}), \lambda_{i+1}(\omega)) = (a_1, \dots, a_i, a_{i+1}).$$

This proves (48). From (38) and (48) we conclude that for all $i \in \mathbb{N}_0$

$$\begin{aligned} \tau_{i,\omega}(a_{1,\omega}, \dots, a_{i,\omega}) &= \tau_i(a_1, \dots, a_i) \\ \varphi_{i,\omega}(a_{1,\omega}, \dots, a_{i,\omega}) &= \varphi_i(a_1, \dots, a_i). \end{aligned}$$

This implies (36) and (37).

□

Next we derive a lower bound for the randomized n-th minimal errors. Analogous to the classical one it uses the average setting with respect to a probability measure on F. However, due to the additional stochastic component, it is somewhat more involved. For the notation of the average case setting we refer to [6], [7].

Lemma 4. *Let ν be a probability measure on F supported by a finite set. Then for all $n \in \mathbb{N}_0$,*

$$e_n^{\mathrm{ran}}(S, F \times \Omega, G) \geq \frac{1}{3} \inf_{D \in \Sigma, \mathbb{P}(D) \geq 1/4} \int_D e_{2n}^{\mathrm{avg}}(S_\omega, \nu, G) d\mathbb{P}(\omega). \tag{50}$$

Proof. Let $A \in \mathscr{A}_n^{\mathrm{ran}}(\mathscr{P})$, $A = ((\Omega_1, \Sigma_1, \mathbb{P}_1),\ (A_{\omega_1})_{\omega_1 \in \Omega_1})$. Then

$$\begin{aligned} n &\geq \sup_{f \in F} \int_{\Omega_1 \times \Omega} \operatorname{card}(A_{\omega_1}, f, \omega) d\mathbb{P}_1(\omega_1) d\mathbb{P}(\omega) \\ &\geq \int_F \int_{\Omega_1 \times \Omega} \operatorname{card}(A_{\omega_1}, f, \omega) d\mathbb{P}_1(\omega_1) d\mathbb{P}(\omega) d\nu(f) \\ &= \int_{\Omega_1 \times \Omega} \int_F \operatorname{card}(A_{\omega_1}, f, \omega) d\nu(f) d\mathbb{P}_1(\omega_1) d\mathbb{P}(\omega). \end{aligned} \tag{51}$$

Let

$$B = \left\{ (\omega_1, \omega) \in \Omega_1 \times \Omega : \int_F \operatorname{card}(A_{\omega_1}, f, \omega) d\nu(f) \leq 2n \right\} \tag{52}$$

and for $\omega_1 \in \Omega_1$, $B_{\omega_1} = \{\omega : (\omega_1, \omega) \in B\}$. Since ν is of finite support, it follows that $B \in \Sigma_1 \times \Sigma$ and $B_{\omega_1} \in \Sigma$. We also set $B' = \{\omega_1 : \mathbb{P}(B_{\omega_1}) \geq 1/4\}$, then $B' \in \Sigma_1$. Moreover, (51) and (52) yield $(\mathbb{P}_1 \times \mathbb{P})(B) \geq 1/2$, hence $\frac{1}{2} \leq \mathbb{P}_1(B') + \frac{1}{4}(1 - \mathbb{P}_1(B'))$, which implies $\mathbb{P}_1(B') \geq 1/3$.

Now we estimate the error of $A = (A_{\omega_1})_{\omega_1 \in \Omega_1}$ from below. For each $\omega_1 \in \Omega_1$ and $\omega \in \Omega$, let $A_{\omega_1, \omega}$ be the respective algorithm for S_ω resulting from A_{ω_1} according to Lemma 3.

$$\begin{aligned} e(S, A, F \times \Omega, G) &= \sup_{f \in F} \int_{\Omega_1 \times \Omega} \|S(f, \omega) - A_{\omega_1}(f, \omega)\|_G d\mathbb{P}_1(\omega_1) d\mathbb{P}(\omega) \\ &\geq \int_{\Omega_1 \times \Omega} \int_F \|S_\omega(f) - A_{\omega_1, \omega}(f)\|_G d\nu(f) d\mathbb{P}_1(\omega_1) d\mathbb{P}(\omega) \\ &\geq \int_{B'} \int_{B_{\omega_1}} \int_F \|S_\omega(f) - A_{\omega_1, \omega}(f)\|_G d\nu(f) d\mathbb{P}(\omega) d\mathbb{P}_1(\omega_1) \\ &\geq \int_{B'} \int_{B_{\omega_1}} e_{2n}^{\mathrm{avg}}(S_\omega, \nu, G) d\mathbb{P}(\omega) d\mathbb{P}_1(\omega_1) \\ &\geq \frac{1}{3} \inf_{D \in \Sigma, \mathbb{P}(D) \geq 1/4} \int_D e_{2n}^{\mathrm{avg}}(S_\omega, \nu, G) d\mathbb{P}(\omega). \end{aligned}$$

□

Let $(\gamma_j)_{j=1}^\infty$ be a sequence of independent standard Gaussian random variables. For $m \in \mathbb{N}$ we set $\mathscr{J}_m = \{1, 2, \ldots, m\}$.

Lemma 5. *There is a constant $c > 0$ such that for all $m \in \mathbb{N}$*

$$\mathbb{P} \left\{ \omega \in \Omega : \min_{\mathscr{J} \subseteq \mathscr{J}_m, |\mathscr{J}| \geq m/2} \left(\sum_{j \in \mathscr{J}} \gamma_j(\omega)^2 \right)^{1/2} \geq c m^{1/2} \right\} \geq 7/8.$$

Proof. Let $k = \lceil m/2 \rceil$ and let $c_0 > 0$ be a constant to be fixed later on. Then

$$
\begin{aligned}
&\mathbb{P}\left\{ \min_{\mathscr{I} \subseteq \mathscr{I}_m, |\mathscr{I}| \geq m/2} \sum_{j \in \mathscr{I}} \gamma_j(\omega)^2 \geq c_0^2 m \right\} \\
&= \mathbb{P}\left\{ \min_{\mathscr{I} \subseteq \mathscr{I}_m, |\mathscr{I}| = k} \sum_{j \in \mathscr{I}} \gamma_j(\omega)^2 \geq c_0^2 m \right\} \\
&\geq 1 - \sum_{\mathscr{I} \subseteq \mathscr{I}_m, |\mathscr{I}| = k} \mathbb{P}\left\{ \sum_{j \in \mathscr{I}} \gamma_j(\omega)^2 < c_0^2 m \right\} \\
&\geq 1 - 2^{2k}\, \mathbb{P}\left\{ \sum_{j=1}^{k} \gamma_j(\omega)^2 < 2c_0^2 k \right\}. \qquad (53)
\end{aligned}
$$

Furthermore, let B_2^k denote the unit ball of $\mathbb{R}^k$, endowed with the Euclidean norm $|\cdot|$. There is a constant $c_1 > 0$ such that for all $k \in \mathbb{N}$

$$
\mathrm{Vol}\left(B_2^k\right) \leq c_1^k k^{-k/2}, \qquad (54)
$$

see, e.g., [11], relation 1.18 on p. 11. Consequently,

$$
\begin{aligned}
\mathbb{P}\left\{ \sum_{j=1}^{k} \gamma_j(\omega)^2 < 2c_0^2 k \right\} &= (2\pi)^{-k/2} \int_{|x| \leq c_0 (2k)^{1/2}} e^{-|x|^2/2} dx \\
&\leq \mathrm{Vol}\left(c_0 (2k)^{1/2} B_2^k\right) = c_0^k (2k)^{k/2} \mathrm{Vol}\left(B_2^k\right) \\
&\leq 2^{k/2} c_0^k c_1^k. \qquad (55)
\end{aligned}
$$

Joining (53) and (55) and setting $c_0 = 2^{-11/2} c_1^{-1}$, we arrive at

$$
\mathbb{P}\left\{ \min_{\mathscr{I} \subseteq \mathscr{I}_m, |\mathscr{I}| \geq m/2} \sum_{j \in \mathscr{I}} \gamma_j(\omega)^2 \geq c_0^2 m \right\} \geq 1 - 2^{-3k} \geq 7/8.
$$

□

Proof of Theorem 3. Two parts of the lower bound estimates easily reduce to known results. Let $\theta_0 \in C(Q)^*$ be defined by $\theta_0(f) = f(0)$ $(f \in C(Q))$. Firstly, we let $f_1(s,t,\omega) = \kappa t$ and put $F_1 := \{f_1\}$. Then $F_1 \subseteq \mathscr{F}_2^r(Q \times [0,1] \times \Omega; \kappa)$. We have

$$
(\theta_0 \circ \mathscr{S}_1)(f_1, \omega) = \kappa \left(\int_0^1 t dW(t) \right)(\omega) \quad (\mathbb{P}\text{-almost surely}).
$$

Consequently,

$$
e_n^{\mathrm{ran}}(\mathscr{S}_1, \mathscr{F}_2^r \times \Omega, C(Q)) \geq e_n^{\mathrm{ran}}(\theta_0 \circ \mathscr{S}_1, F_1 \times \Omega, \mathbb{R}) \geq cn^{-1}, \qquad (56)
$$

where the last relation follows from Theorem 1 in [15] (who considered deterministic algorithms) and Lemma 2 above. Secondly, we set $f_2(s,t,\omega) \equiv \kappa$ and $F_2 := \{f_2\}$. We have

$$F_2 \subseteq \mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa), \quad F_2 \subseteq F_q^{r_1,\varrho}(Q \times [0,1] \times \Omega; \kappa),$$

and

$$\big((\theta_0 \circ \mathscr{S})(f_2,\omega)\big)(t) = \kappa W(t,\omega).$$

Therefore we get from [14] and [9], using Lemma 2 again,

$$\begin{aligned} & e_n^{\mathrm{ran}}(\mathscr{S}, \mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa) \times \Omega, C(Q \times [0,1])) \\ & \geq \ e_n^{\mathrm{ran}}(\theta_0 \circ \mathscr{S}, F_2 \times \Omega, C([0,1])) \geq c n^{-1/2} (\log n)^{1/2}, \end{aligned} \tag{57}$$

and similarly

$$e_n^{\mathrm{ran}}(\mathscr{S}, F_q^{r_1,\varrho}(Q \times [0,1] \times \Omega; \kappa), C(Q \times [0,1])) \ \geq \ c n^{-1/2} (\log n)^{1/2}. \tag{58}$$

Now we consider a third subclass. Let φ_0 be a C^∞ function on $\mathbb{R}^d$ with support in Q and $\|\varphi_0\|_{C(Q)} = 1$ and let φ_1 be a C^∞ function on $\mathbb{R}$ with support in $[0,1]$ and $\|\varphi_1\|_{L_2(\mathbb{R})} = 1$. Let $m_0, m_1 \in \mathbb{N}$ and let Q_i $(i = 1, \dots, m_0^d)$ be the subdivision of Q into m_0^d cubes of disjoint interior of side-length m_0^{-1}. Let s_i be the point in Q_i with minimal coordinates. Put $t_j = j/m_1$ and define for $s \in Q$, $t \in [0,1]$, $i = 1, \dots, m_0^d$, $j = 1, \dots, m_1$

$$\varphi_{0,i}(s) = \varphi_0(m_0(s - s_i)), \quad \varphi_{1,j}(t) = \varphi_1(m_1(t - t_j)), \quad \psi_{ij}(s,t) = \varphi_{0,i}(s)\varphi_{1,j}(t).$$

Denote $\mathscr{K}_{m_0 m_1} = \{1, \dots, m_0^d\} \times \{1, \dots, m_1\}$ and

$$\Psi_{m_0 m_1} = \left\{ \sum_{(i,j) \in \mathscr{K}_{m_0 m_1}} \delta_{ij} \psi_{ij} : \ \delta_{ij} \in \{-1, 0, 1\} \right\}.$$

The stochastic integral $m_1^{1/2} \int_0^1 \varphi_{1,j}(t) dW(t)$ is an element of $L_2(\Omega)$, hence an equivalence class. Let the function $\gamma_j = \gamma_j(\omega)$ be any representative of it. Since $\|\varphi_{1,j}\|_{L_2([0,1])} = m_1^{-1/2}$, the $(\gamma_j)_{j=1}^{m_1}$ are independent standard Gaussian random variables. By (20) and the linearity of the stochastic integral we have for $(i,j) \in \mathscr{K}_{m_0 m_1}$ and each $s \in Q$

$$(\mathscr{S}_1(\psi_{ij}, \omega))(s) \ = \ m_1^{-1/2} \varphi_{0,i}(s) \gamma_j(\omega)$$

$\mathbb{P}$-almost surely. Using continuity and a density argument yields that there is an $\Omega_0 \in \Sigma$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ and $(i,j) \in \mathscr{K}_{m_0 m_1}$

$$\mathscr{S}_1(\psi_{ij}, \omega) \ = \ m_1^{-1/2} \varphi_{0,i} \gamma_j(\omega).$$

We conclude, using the linearity of $\mathscr{S}_1$ that for all $\delta_{ij}\in\{-1,0,1\}$ and $\omega\in\Omega_0$

$$\mathscr{S}_1\Bigg(\sum_{(i,j)\in\mathscr{K}_{m_0m_1}}\delta_{ij}\psi_{ij},\omega\Bigg) \;=\; m_1^{-1/2}\sum_{i=1}^{m_0^d}\varphi_{0,i}\sum_{j=1}^{m_1}\delta_{ij}\gamma_j(\omega). \tag{59}$$

Let $\{\varepsilon_{ij}:(i,j)\in\mathscr{K}_{m_0m_1}\}$ be independent Bernoulli random variables with $\mathbb{P}_2\{\varepsilon_{ij}=-1\}=\mathbb{P}_2\{\varepsilon_{ij}=+1\}=1/2$ on a probability space $(\Omega_2,\Sigma_2,\mathbb{P}_2)$. Let ν be the distribution of the $\Psi_{m_0m_1}$ valued random variable $\sum_{(i,j)\in\mathscr{K}_{m_0m_1}}\varepsilon_{ij}\psi_{ij}$. Let $n\in\mathbb{N}$ be such that

$$m_0^dm_1\ge 8n. \tag{60}$$

According to Lemma 4

$$e_n^{\rm ran}(\mathscr{S}_1,\Psi_{m_0m_1}\times\Omega,C(Q))\ge\frac{1}{3}\inf_{D\in\Sigma,\mathbb{P}(D)\ge 1/4}\int_D e_{2n}^{\rm avg}(\mathscr{S}_{1,\omega},\nu,C(Q))d\mathbb{P}(\omega). \tag{61}$$

Lemma 6 of [6] together with (59) and (60) gives for $\omega\in\Omega_0$

$$\begin{aligned}
&e_{2n}^{\rm avg}(\mathscr{S}_{1,\omega},\nu,C(Q))\\
&\ge \frac{1}{2}\min_{\mathscr{K}\subseteq\mathscr{K}_{m_0m_1},|\mathscr{K}|\ge m_0^dm_1-4n}\mathbb{E}_{\mathbb{P}_2}\left\|\mathscr{S}_1\Bigg(\sum_{(i,j)\in\mathscr{K}}\varepsilon_{ij}\psi_{ij},\omega\Bigg)\right\|_{C(Q)}\\
&\ge \frac{1}{2}\min_{\mathscr{K}\subseteq\mathscr{K}_{m_0m_1},|\mathscr{K}|\ge m_0^dm_1-4n}\mathbb{E}_{\mathbb{P}_2}\left\|m_1^{-1/2}\sum_{i=1}^{m_0^d}\varphi_{0,i}\sum_{j:\,(i,j)\in\mathscr{K}}\varepsilon_{ij}\gamma_j(\omega)\right\|_{C(Q)}\\
&\ge \frac{1}{2}m_1^{-1/2}\min_{\mathscr{K}\subseteq\mathscr{K}_{m_0m_1},|\mathscr{K}|\ge m_0^dm_1-4n}\mathbb{E}_{\mathbb{P}_2}\max_{1\le i\le m_0^d}\left|\sum_{j:\,(i,j)\in\mathscr{K}}\varepsilon_{ij}\gamma_j(\omega)\right|\\
&\ge \frac{1}{2}m_1^{-1/2}\min_{\mathscr{K}\subseteq\mathscr{K}_{m_0m_1},|\mathscr{K}|\ge m_0^dm_1-4n}\max_{1\le i\le m_0^d}\mathbb{E}_{\mathbb{P}_2}\left|\sum_{j:\,(i,j)\in\mathscr{K}}\varepsilon_{ij}\gamma_j(\omega)\right|.
\end{aligned}\tag{62}$$

By Khintchine's inequality, see [10], Th. 2.b.3,

$$\mathbb{E}_{\mathbb{P}_2}\left|\sum_{j:\,(i,j)\in\mathscr{K}}\varepsilon_{ij}\gamma_j(\omega)\right|\ge c\Bigg(\sum_{j:\,(i,j)\in\mathscr{K}}\gamma_j(\omega)^2\Bigg)^{1/2}.$$

So we obtain from (62)

$$\begin{aligned}
&e_{2n}^{\rm avg}(\mathscr{S}_{1,\omega},\nu,C(Q))\\
&\ge cm_1^{-1/2}\min_{\mathscr{K}\subseteq\mathscr{K}_{m_0m_1},|\mathscr{K}|\ge m_0^dm_1-4n}\max_{1\le i\le m_0^d}\Bigg(\sum_{j:\,(i,j)\in\mathscr{K}}\gamma_j(\omega)^2\Bigg)^{1/2}.
\end{aligned}$$

For each $\mathscr{K} \subseteq \mathscr{K}_{m_0 m_1}$ with $|\mathscr{K}| \geq m_0^d m_1 - 4n$ we have by (60) $|\mathscr{K}| \geq |\mathscr{K}_{m_0 m_1}|/2$, hence there is an i with $1 \leq i \leq m_0^d$ such that $|\{j : (i,j) \in \mathscr{K}\}| \geq m_1/2$. With $\mathscr{J}_{m_1} = \{1, 2, \ldots, m_1\}$ it follows that

$$e_{2n}^{\mathrm{avg}}(\mathscr{S}_{1,\omega}, \nu, C(Q)) \geq c m_1^{-1/2} \min_{\mathscr{J} \subseteq \mathscr{J}_{m_1}, |\mathscr{J}| \geq m_1/2} \left(\sum_{j \in \mathscr{J}} \gamma_j(\omega)^2 \right)^{1/2},$$

and therefore, by (61) and Lemma 5, for n satisfying (60),

$$\begin{aligned} & e_n^{\mathrm{ran}}(\mathscr{S}_1, \Psi_{m_0 m_1} \times \Omega, C(Q)) \\ \geq \; & c m_1^{-1/2} \inf_{D \in \Sigma, \mathbb{P}(D) \geq 1/4} \int_D \min_{\mathscr{J} \subseteq \mathscr{J}_{m_1}, |\mathscr{J}| \geq m_1/2} \left(\sum_{j \in \mathscr{J}} \gamma_j(\omega)^2 \right)^{1/2} d\mathbb{P}(\omega) \geq c. \quad (63) \end{aligned}$$

Let $2 \leq q < \infty$ and observe that there is a constant $c_0 > 0$ such that for $m_0, m_1 \in \mathbb{N}$

$$c_0 (\max(m_0, m_1))^{-r} \Psi_{m_0 m_1} \subseteq \mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa). \quad (64)$$

For $n \in \mathbb{N}$ put $m_0 = m_1 = \left\lceil 4 n^{\frac{1}{d+1}} \right\rceil$, hence (60) is satisfied, and therefore (18), (63), and (64) imply

$$\begin{aligned} & e_n^{\mathrm{ran}}(\mathscr{S}, \mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa) \times \Omega, C(Q \times [0,1])) \\ \geq \; & e_n^{\mathrm{ran}}(\mathscr{S}_1, \mathscr{F}_q^r(Q \times [0,1] \times \Omega; \kappa) \times \Omega, C(Q)) \\ \geq \; & c_0^{-1} m_0^{-r} e_n^{\mathrm{ran}}(\mathscr{S}_1, \Psi_{m_0 m_1} \times \Omega, C(Q)) \geq c n^{-\frac{r}{d+1}}. \quad (65) \end{aligned}$$

Combining (56–57) and (65) proves the lower bounds (31) and (33).

Next let $2 < q < \infty$ and note that there is a constant $c_1 > 0$ such that for all $m_0, m_1 \in \mathbb{N}$

$$c_1 m_0^{-r_1} m_1^{-\varrho} \Psi_{m_0 m_1} \subseteq F_q^{r_1, \varrho}(Q \times [0,1] \times \Omega; \kappa) \subseteq F^{r_1, \varrho}(Q \times [0,1] \times \Omega; \kappa). \quad (66)$$

Let $n \in \mathbb{N}$. First we put $m_0 = \left\lceil 8 n^{\frac{1}{d}} \right\rceil$, $m_1 = 1$. Again (60) is satisfied, thus (18), (63), and (66) yield

$$\begin{aligned} & e_n^{\mathrm{ran}}(\mathscr{S}, F_q^{r_1, \varrho} \times \Omega, C(Q \times [0,1])) \geq e_n^{\mathrm{ran}}(\mathscr{S}_1, F_q^{r_1, \varrho} \times \Omega, C(Q)) \\ \geq \; & c_1^{-1} m_0^{-r} e_n^{\mathrm{ran}}(\mathscr{S}_1, \Psi_{m_0 m_1} \times \Omega, C(Q)) \geq c n^{-\frac{r}{d}}. \quad (67) \end{aligned}$$

Now we set $m_0 = 1$, $m_1 = 8n$. Clearly, (60) holds and, using again (18), (63), and (66), we conclude

$$\begin{aligned} & e_n^{\mathrm{ran}}(\mathscr{S}, F_q^{r_1, \varrho} \times \Omega, C(Q \times [0,1])) \geq e_n^{\mathrm{ran}}(\mathscr{S}_1, F_q^{r_1, \varrho} \times \Omega, C(Q)) \\ \geq \; & c_1^{-1} m_1^{-\varrho} e_n^{\mathrm{ran}}(\mathscr{S}_1, \Psi_{m_0 m_1} \times \Omega, C(Q)) \geq c n^{-\varrho}. \quad (68) \end{aligned}$$

Now the lower bounds (32) and (34) follow from (58) and (67–68), which completes the proof. □

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