Complexity of Banach space valued and parametric stochastic Itô integration

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Dedicated to the memory of Joseph F. Traub

Abstract

We present a complexity analysis for strong approximation of Banach space valued and parameter dependent scalar stochastic Itô integration, driven by a Wiener process. Both definite and indefinite integration are considered. We analyze the Banach space valued version of the Euler-Maruyama scheme. Based on these results, we define a multilevel algorithm for the parameter dependent stochastic integration problem and show its order optimality for various input classes.

1 Introduction

The complexity of stochastic integration was first investigated in [22]. The authors consider the problem of approximating stochastic Itô integrals of the form $\int_0^1 f(t, W(t))dW(t)$, where $(W(t))_{t\in[0,1]}$, $W(t) = W(t, \omega)$, denotes a standard Wiener process on a probability space $(\Omega, \Sigma, \mathbb{P})$. They studied the complexity of the problem, which depends on the smoothness of the integration function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$. For this purpose, they analyzed the Milstein scheme and provided a matching lower bound for certain problem classes. Moreover, they analyzed the Euler-Maruyama scheme and conjectured its order optimality (for certain problem classes). This conjecture was later proved to be true in [10]. The results are based on the assumption that standard information is available, i.e., evaluations of f and $W(t)$. The case of linear information was investigated in [17].

Extending the analysis of [22], we study the complexity of definite and indefinite stochastic Itô integration of random functions $f : [0, 1] \times \Omega \to X$, with X a Banach space, thus, with $f(t) = f(t, \omega)$ we are interested in the approximation of

$$\int_0^t f(\tau)dW(\tau)$$

for all $t \in [0,1]$ simultaneously in the indefinite case, and respectively for $t = 1$ in the definite case. Stochastic integration in X is closely connected to a geometric property of X, namely, the martingale type 2. We go beyond this class by considering functions Tf which are images of functions with values in some Banach space Y under an operator $T : Y \to X$ of martingale type 2. This is needed for our second goal in this paper: the investigation of stochastic Itô integration of parameter dependent scalar valued functions $f : Q \times [0,1] \times \Omega \to \mathbb{R}$,

$$\int_0^t f(s,\tau)dW(\tau) \quad (s \in Q, t \in [0,1]),$$

where $f(s,\tau) = f(s,\tau,\omega)$ and $Q = [0,1]^d$ is the parameter domain.

The complexity of Banach space valued (non-stochastic) integration was first considered in [2], the complexity of (non-stochastic) parametric integration has been treated in [9],[7],[8], and also in [2]. It turned out that the consideration of Banach space valued algorithms can be crucial for the analysis of parametric problems. Here we follow the same line to derive algorithms for parametric stochastic integration and state complexity results. We define and analyze the Banach space valued versions of the Euler-Maruyama scheme. We obtain the same order of convergence as for the scalar valued case.

A similar situation occurs in the case of parameter dependent stochastic integration, where two cases have to be distinguished. In the case of higher parameter smoothness we obtain the same rate (up to logarithmic factors) as for non-parametric scalar stochastic integration. In the case of lower parameter smoothness we obtain the rate (again up to logarithmic factors) of approximation of functions depending only on the parameter – in other words, a rate as if we had full knowledge on the integrals. These improvements are achieved due to the multilevel structure of the algorithms.

The multilevel technique, used here, was first introduced for the complexity analysis in the randomized setting of problems such as global solution of integral equations in [6] and parametric integration in [9], see also [7],[8]. Later such multilevel schemes were used for the approximation of quadrature problems of stochastic differential equations, see [5]. Our general multilevel algorithm is a combination of a Banach space valued algorithm and common interpolation operators connected via multilevel techniques.

We also prove lower bounds which are matching with the upper bounds resulting from the error estimates, thus showing the optimality of the algorithms and establishing the complexity of the considered problems (in some cases up to logarithmic factors).

The structure of the paper is as follows: In the second section, we briefly introduce the needed results from probability theory and stochastic integration in Banach spaces. In the third section, we analyze the Euler-Maruyama scheme for Banach space valued stochastic integrals, while in Section 4, we develop a general multilevel scheme in Banach spaces which is similar to the one introduced in [2] and [3]. We apply this algorithm to the scalar parametric case in Section 5 and finally, in Section 6, we present the complexity results for the previously considered problems.

2 Preliminaries

2.1 Notation

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Let X, Y be Banach spaces. The closed unit ball of X is denoted by B_X, the dual space by X^*, the identity operator on X by I_X, and the space of bounded linear operators from Y to X by $\mathscr{L}(Y, X)$. Let $d \in \mathbb{N}$. The space of continuous functions on a compact set $Q \subset \mathbb{R}^d$ with values in X is denoted by $C(Q, X)$ and is equipped with the supremum norm. Furthermore, if Q is the closure of an open bounded set, then for $r \in \mathbb{N}$, $C^r(Q, X)$ stands for the space of all functions $f : Q \to X$ which are r-times continuously differentiable (with respect to the norm topology of X) in the interior of Q and which together with their derivatives up to order r are bounded and possess continuous extensions to all of Q. This space is equipped with the norm

$$\|f\|_{C^r(Q,X)} = \sup_{|\alpha|\le r,\, s\in Q} \left\| \frac{\partial^{|\alpha|} f(s)}{\partial s^{\alpha}} \right\|_X$$

with $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $|\alpha| = |\alpha_1| + \cdots + |\alpha_d|$. If $r = 0$, we set $C^0(Q, X) = C(Q, X)$. For $0 \le \varrho \le 1$ we let $\mathscr{C}^{\varrho}(Q, X)$ be the space of all $f \in C(Q, X)$ satisfying

$$\|f\|_{\mathscr{C}^{\varrho}(Q,X)} \quad := \quad \max\left(\|f\|_{C(Q,X)}, \sup_{s\neq t\in Q} |s-t|^{-\varrho} \left\|f(s) - f(t)\right\|_X \right) < \infty,$$

For $1 \le p < \infty$ and $(D, \mathscr{D}, \nu)$ an arbitrary measure space, $L_p(D, \mathscr{D}, \nu, X)$ is the space of (equivalence classes of) X-valued Bochner p-integrable functions on D, equipped with the usual norm. Note that, by definition, each function in $L_p(D, \mathscr{D}, \nu, X)$ is, except for a set of ν-measure zero, the pointwise limit of a sequence of simple functions. Consequently, for each $f \in L_p(D, \mathscr{D}, \nu, X)$ there is a separable subspace X_0 of X such that f takes values in X_0, except for a set of ν-measure zero. If there is no ambiguity about $(D, \mathscr{D}, \nu)$ we skip $\mathscr{D}$ and/or ν. If $X = \mathbb{R}$, we skip $\mathbb{R}$ in the notation above and write $C^r(Q)$, $\mathscr{C}^{\varrho}(Q)$, $L_p(D, \mathscr{D}, \nu)$ etc.

Throughout the paper the same symbol $c, c_1, c_2, \dots$ may denote different constants, even in a sequence of relations. Moreover, for nonnegative reals $(a_n)_{n\in\mathbb{N}}$

and $(b_n)_{n\in\mathbb{N}}$ we write $a_n \preceq b_n$ if there are constants $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $a_n \leq cb_n$. Furthermore, $a_n \asymp b_n$ means that $a_n \preceq b_n$ and $b_n \preceq a_n$.

Let $Q = [0,1]^d$. Next we introduce some interpolation operators on Q. For $k \in \mathbb{N}$, let $\Gamma_k^d = \left\{\frac{i}{k} : 0 \leq i \leq k\right\}^d$ be the uniform d-dimensional grid of meshsize $1/k$ in Q. For $r, m \in \mathbb{N}$ let $P_m^{r,1} \in \mathscr{L}(C([0,1]))$ be composite Lagrange interpolation of degree r with respect to the partition of $[0,1]$ given by Γ_m^1 and

$$P_m^{r,d} = \otimes^d P_m^{r,1} \in \mathscr{L}(C([0,1]^d))$$

its d-fold tensor product. Let

$$P_m^{r,d} f = \sum_{s\in\Gamma_{rm}^d} f(s)\varphi_{m,s}^{r,d} \quad (f \in C(Q)) \tag{1}$$

be the representation of $P_m^{r,d}$ with $\varphi_{m,s}^{r,d} \in C(Q)$ $(s \in \Gamma_{rm}^d)$,

$$\varphi_{m,s}^{r,d}(s) = 1, \quad \varphi_{m,s}^{r,d}(t) = 0 \quad (t \in \Gamma_{rm}^d, t \neq s). \tag{2}$$

The X-valued versions of the operators above are defined as

$$P_m^{r,d,X} f = \sum_{s\in\Gamma_{rm}^d} f(s)\varphi_{m,s}^{r,d} \quad (f \in C(Q,X)). \tag{3}$$

We will consider $P_m^{r,d,X}$ also as an operator from $\ell_\infty(\Gamma_{rm}^d, X)$ to $C(Q,X)$ in the obvious way. There are constants $c_0, c_1 > 0$ such that for all Banach spaces X and $m \in \mathbb{N}$

$$\|P_m^{r,d,X}\|_{\mathscr{L}(C(Q,X))} \leq c_0 \tag{4}$$

$$\|J^{r,X} - P_m^{r,d,X} J^{r,X}\|_{\mathscr{L}(C^r(Q,X),C(Q,X))} \leq c_1 m^{-r}, \tag{5}$$

where $J^{r,X} : C^r(Q,X) \to C(Q,X)$ is the embedding. The scalar case is well-known (see, e.g., [1], Th. 3.1.4). For the simple derivation of the Banach space case from the scalar case we refer to [2], Section 2.

2.2 Stochastic integration in Banach spaces

We introduce some concepts from probability theory in Banach spaces needed in the sequel. Let $\mathscr{B}(X)$ be the σ-algebra of Borel subsets of X, that is, the smallest σ-algebra containing the norm-open sets. A sequence $(\eta_i)_{i=1}^n \subset L_2(\Omega,\Sigma,\mathbb{P},X)$ is called an X-valued martingale difference sequence, if there is an X-valued martingale $(M_i)_{i=0}^n \subset L_2(\Omega,\Sigma,\mathbb{P},X)$ with $M_0 \equiv 0$ and $\eta_i = M_i - M_{i-1}$. The martingale type 2 constant $\mu_2(T)$ of an operator $T \in \mathscr{L}(Y,X)$ is defined as the smallest constant $c > 0$ such that for all probability spaces $(\Omega,\Sigma,\mathbb{P})$, all $n \in \mathbb{N}$, and martingale difference sequences $(\eta_i)_{i=1}^n \subset L_2(\Omega,\Sigma,\mathbb{P},Y)$,

$$\mathbb{E}\left\|\sum_{i=1}^n T\eta_i\right\|_X^2 \leq c^2 \sum_{i=1}^n \mathbb{E}\|\eta_i\|_Y^2.$$

We say that T has martingale type 2 if $\mu_2(T) < \infty$. The space of all operators from Y to X of martingale type 2 is denoted by $\mathscr{M}_2(Y, X)$. Clearly, $\|T\| \le \mu_2(T)$, and endowed with the μ_2-norm, $\mathscr{M}_2(Y, X)$ is a Banach space. If $X = Y$ and $T = I_X$, we just write $\mu_2(X)$ instead of $\mu_2(I_X)$, which is called the martingale type 2 constant of the space X. Correspondingly, we say that X has martingale type 2 if $\mu_2(X) < \infty$.

All finite dimensional Banach spaces have martingale type 2. If X is 2-smooth, then it has martingale type 2 (see Pisier [13] and [14], ch. 6, for notation and this result). It follows that for $2 \le p < \infty$ the space $L_p(D, \mathscr{D}, \nu)$, with $(D, \mathscr{D}, \nu)$ an arbitrary measure space, has martingale type 2, see also the proof of Lemma 2.1 below.

For our analysis we need the following result, essentially contained in [15]. For the sake of completeness, and since we did not find a direct reference for the estimate (6), we include the short proof.

Lemma 2.1. *There is a constant $c > 0$ such that for all $n \in \mathbb{N}, n \ge 2$*

$$\mu_2(\ell_\infty^n) \le c\sqrt{\log n}. \tag{6}$$

Proof. First we show that there is a constant $c > 0$ such that for all p with $2 \le p < \infty$ and all measure spaces $(D, \mathscr{D}, \nu)$

$$\mu_2(L_p(D, \mathscr{D}, \nu)) \le c\sqrt{p}. \tag{7}$$

Let M_n, η_n be as above, with $X = L_p(D, \mathscr{D}, \nu)$. We follow the lines of the proof of Theorem 10.22 in [16] (see also [15], Theorem 4.21). We have

$$\begin{aligned} &\left(\frac{1}{2}\big(\|M_{n-1} + \eta_n\|_{L_p}^2 + \|M_{n-1} - \eta_n\|_{L_p}^2\big)\right)^{1/2} \\ \le\; &\left(\frac{1}{2}\big(\|M_{n-1} + \eta_n\|_{L_p}^p + \|M_{n-1} - \eta_n\|_{L_p}^p\big)\right)^{1/p} \\ \le\; &\left(\|M_{n-1}\|_{L_p}^2 + (p-1)\|\eta_n\|_{L_p}^2\right)^{1/2}, \end{aligned}$$

where the latter estimate follows from (10.35) of Lemma 10.34 in [16] by duality, or use relation (10.37) there, directly (see, respectively, (4.34), Lemma 4.32, and (4.36) in [15]). We conclude

$$\begin{aligned} &\frac{1}{2}\big(\mathbb{E}\|M_n\|_{L_p}^2 + \mathbb{E}\,\|M_{n-1}\|_{L_p}^2\big) \\ \le\; &\frac{1}{2}\big(\mathbb{E}\|M_n\|_{L_p}^2 + \mathbb{E}\,\|M_{n-1} - \eta_n\|_{L_p}^2\big) \le \mathbb{E}\|M_{n-1}\|_{L_p}^2 + (p-1)\mathbb{E}\|\eta_n\|_{L_p}^2, \end{aligned}$$

thus

$$\mathbb{E}\|M_n\|_{L_p}^2 \le \mathbb{E}\|M_{n-1}\|_{L_p}^2 + 2(p-1)\mathbb{E}\|\eta_n\|_{L_p}^2.$$

By recursion

$$\mathbb{E}\|M_n\|_{L_p}^2 \leq 2(p-1)\sum_{i=1}^n \mathbb{E}\|\eta_i\|_{L_p}^2,$$

which shows (7). Now let $n \in \mathbb{N}$, $n \geq 4$ (for $n < 4$ relation (6) is a consequence of the elementary scalar case $\mu_2(\mathbb{R}) = 1$) and put $p = \log_2 n$. Let $J : \ell_\infty^n \mapsto \ell_p^n$ be the identity. Then

$$\|J\| = n^{1/p} = 2, \qquad \|J^{-1}\| = 1.$$

This together with (7) shows (6). □

Now we introduce the needed notions of Banach space valued stochastic integrals. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, $\mathscr{F} = (\mathscr{F}_t)_{0\leq t<\infty}$, $\mathscr{F}_t \subseteq \Sigma$ a filtration, let $W = (W(t))_{0\leq t<\infty}$, $W(t) = W(t,\omega)$ be a Wiener process on $(\Omega, \Sigma, \mathbb{P})$, adapted to $\mathscr{F}$ and such that for $0 \leq t_1 \leq t_2$ the increments $W(t_2)-W(t_1)$ are independent of $\mathscr{F}_{t_1}$. Moreover, we assume that

$$\Omega_0 := \{\omega : W(\,\cdot\,,\omega) \text{ is continuous}\} \in \Sigma, \quad \mathbb{P}(\Omega_0) = 1. \tag{8}$$

Let $0 \leq a < b < \infty$ and let $\mathscr{B}\mathscr{F}$ denote the σ-algebra of progressively measurable subsets of $[a,b]\times\Omega$, that is, all subsets B with the property that for each $\tau \in [a,b]$

$$B \cap [a,\tau]\times\Omega \in \mathscr{B}([a,\tau]) \times \mathscr{F}_\tau \quad (\tau \in [a,b])$$

Let mes denote the Lebesgue measure on $[a,b]$. We consider the space $L_2([a,b]\times\Omega, \mathscr{B}\mathscr{F}, \text{mes}\times\mathbb{P}, X)$, which consists of (equivalence classes) of square-integrable progressively measurable functions $f : [a,b]\times\Omega \to X$, meaning that $f|_{[a,\tau]\times\Omega}$ is $\mathscr{B}([a,\tau]) \times \mathscr{F}_\tau$-to-$\mathscr{B}(X)$ measurable for all $\tau \in [a,b]$.

A function $g : [a,b]\times\Omega \to X$ is called a progressively measurable step function, if there are $k \in \mathbb{N}$, points $(\tau_i)_{i=0}^k \subset [a,b]$, $\tau_0 < \cdots < \tau_k$, and $g_i \in L_2(\Omega, \mathscr{F}_{\tau_i}, X)$ such that

$$g(t) = \sum_{i=0}^{k-1} g_i \chi_{(\tau_i,\tau_{i+1}]}(t) \qquad (t \in [a,b]). \tag{9}$$

For such a function we define the stochastic integral

$$\int_a^b g(t)dW(t) \in L_2(\Omega, X)$$

by setting

$$\int_a^b g(t)dW(t) = \sum_{i=0}^{k-1} g_i(W(\tau_{i+1}) - W(\tau_i)). \tag{10}$$

Following Dettweiler [4], we say that a function $f \in L_2([a,b]\times\Omega, \mathscr{B}\mathscr{F}, X)$ is stochastically integrable with respect to W, if there is a sequence of progressively measurable step functions f_n such that

$$\begin{aligned}\lim_{n\to\infty} \sup_{z\in B_{X^*}} \|\langle f, z\rangle - \langle f_n, z\rangle\|_{L_2([a,b]\times\Omega)} &= 0, \\ \lim_{m,n\to\infty} \left\| \int_a^b f_m(t)dW(t) - \int_a^b f_n(t)dW(t) \right\|_{L_2(\Omega,X)} &= 0.\end{aligned}$$

The space of all such functions f is denoted by $L_2([a,b]\times\Omega, \mathscr{BF}, W, X)$. The stochastic integral of $f \in L_2([a,b]\times\Omega, \mathscr{BF}, W, X)$ is then defined as the limit in $L_2(\Omega, X)$ of the integrals of the functions f_n,

$$\int_a^b f(t)dW(t) = \lim_{n\to\infty} \int_a^b f_n(t)dW(t).$$

It follows from the definition that if $z \in X^*$, then

$$\left\langle \int_a^b f(t)dW(t), z \right\rangle = \int_a^b \langle f(t), z\rangle \, dW(t). \tag{11}$$

Note that X is of martingale type 2, if and only if

$$L_2([a,b]\times\Omega, \mathscr{BF}, X) \subset L_2([a,b]\times\Omega, \mathscr{BF}, W, X),$$

see [4]. For our applications to parametric stochastic integration the class of spaces of martingale type 2 is too narrow, since (as in [9] and [2]) we want to study the problem with error measured in the maximum norm. That is, as target spaces we consider spaces of continuous functions – which are not of martingale type 2. We therefore use an operator approach, also developed in [4], see section 5 there.

Let X, Y be Banach spaces, let $T \in \mathscr{M}_2(Y, X)$, and let $f \in L_2([a,b]\times\Omega, \mathscr{BF}, Y)$. We first show that $Tf \in L_2([a,b]\times\Omega, \mathscr{BF}, W, X)$. Indeed, let $g \in L_2([a,b]\times\Omega, Y)$ be a progressively measurable step function with representation (9). Then we have

$$\begin{aligned}\left\| \int_a^b Tg(t)dW(t) \right\|_{L_2(\Omega,X)} &= \left(\mathbb{E} \left\| \sum_{i=0}^{k-1} Tg_i(W(\tau_{i+1}) - W(\tau_i)) \right\|_X^2 \right)^{1/2} \\ &\le \mu_2(T) \left(\mathbb{E} \sum_{i=0}^{k-1} \|g_i\|_Y^2 (W(\tau_{i+1}) - W(\tau_i))^2 \right)^{1/2} \\ &= \mu_2(T) \|g\|_{L_2([a,b]\times\Omega,Y)}. \end{aligned} \tag{12}$$

Now let $(f_n)_{n=1}^\infty \subset L_2([a,b]\times\Omega, Y)$ be a sequence of progressively measurable step functions such that

$$\lim_{n\to\infty} \|f - f_n\|_{L_2([a,b]\times\Omega,Y)} = 0. \tag{13}$$

One could take, e.g.,

$$f_n(t,\omega) = n\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i} f(\tau,\omega)d\tau\,\chi_{(t_i,t_{i+1}]}(t) \tag{14}$$

with $t_i = a + \frac{i(b-a)}{n}$. It follows that

$$\lim_{n\to\infty}\sup_{z\in B_{X^*}} \|\langle Tf,z\rangle - \langle Tf_n,z\rangle\|_{L_2([a,b]\times\Omega)} \;=\; 0$$

and from (12)

$$\lim_{m,n\to\infty}\left\|\int_a^b Tf_m(t)dW(t) - \int_a^b Tf_n(t)dW(t)\right\|_{L_2(\Omega,X)} \;=\; 0.$$

Thus, $Tf \in L_2([a,b]\times\Omega,\mathscr{BF},W,X)$ and

$$\int_a^b Tf(t)dW(t) = \lim_{n\to\infty}\int_a^b Tf_n(t)dW(t), \tag{15}$$

in the norm of $L_2(\Omega,X)$. Moreover, applying (12) to f_n and passing to the limit gives

$$\left(\mathbb{E}\left\|\int_a^b Tf(t)dW(t)\right\|_X^2\right)^{1/2} \le \mu_2(T)\left(\int_a^b \mathbb{E}\,\|f(t)\|_Y^2 dt\right)^{1/2}. \tag{16}$$

It follows from (15) that the indefinite stochastic integral, that is, the stochastic integral with variable upper limit

$$\left(\int_a^t Tf(\tau)dW(\tau)\right)_{t\in[a,b]} \tag{17}$$

is a martingale. We claim that there is a version of (17) whose trajectories are continuous, that is, in $C([a,b],X)$. Indeed, let $f_n : [a,b]\times\Omega\to X$ be as above, see (13),

$$f_n(t,\omega) = \sum_{i=0}^{k_n-1} f_{ni}(\omega)\chi_{(\tau_{ni},\tau_{n,i+1}]}(t) \qquad (t\in[a,b],\omega\in\Omega),$$

with $f_{ni}\in L_2(\Omega,X)$. We can assume without loss of generality that $\tau_{n,0}=a$, $\tau_{n,n}=b$ for all n. Define a function $h_n : [a,b]\times\Omega\to X$ by setting

$$h_n(t,\omega) = 0 \quad \text{if } t=0 \text{ or } \omega\notin\Omega_0, \tag{18}$$

where Ω_0 was defined in (8), and for $\omega\in\Omega_0$, $t\in(\tau_{nj},\tau_{n,j+1}]$ $(j=0,\dots,k_n-1)$

$$\begin{aligned} h_n(t,\omega) \;&=\; \sum_{i<j} Tf_{ni}(\omega)(W(\tau_{n,i+1},\omega) - W(\tau_{ni},\omega)) \\ &\qquad + Tf_{nj}(\omega)(W(t,\omega) - W(\tau_{nj},\omega)). \end{aligned} \tag{19}$$

By (8), $h_n(\,\cdot\,,\omega) \in C([a,b],X)$ for all $\omega \in \Omega$. We define a function

$$\bar{h}_n : \Omega \to C([a,b],X), \quad \bar{h}_n(\omega) = h_n(\,\cdot\,,\omega).$$

Then $\bar{h}_n \in L_2(\Omega, C([a,b],X))$. For $t \in [a,b]$ let

$$\delta_t \in \mathscr{L}(C([a,b],X),X), \quad \delta_t g = g(t).$$

We have

$$\delta_t \bar{h}_n = \int_a^t T f_n(\tau) dW(\tau). \tag{20}$$

Since $h_n(t,\omega)$ is a continuous martingale, we obtain by the Kolmogorov-Doob inequality and (16),

$$\begin{aligned}
&\left\|\bar{h}_m - \bar{h}_n\right\|^2_{L_2(\Omega,C([a,b],X))} \\
&= \mathbb{E} \sup_{t\in[a,b]} \|h_m(t,\omega) - h_n(t,\omega)\|_X^2 \le 4\mathbb{E}\, \|h_m(b,\omega) - h_n(b,\omega)\|_X^2 \\
&= 4\mathbb{E} \left\| \int_a^b T f_m(\tau) dW(\tau) - \int_a^b T f_n(\tau) dW(\tau) \right\|_X^2 \\
&\le 4\mu_2(T)\, \|f_m - f_n\|_{L_2([a,b]\times\Omega,Y)} \to 0
\end{aligned} \tag{21}$$

as $m,n \to \infty$. Hence, there is a function $\bar{h} \in L_2(\Omega, C([a,b],X))$ with

$$\left\|\bar{h} - \bar{h}_n\right\|_{L_2(\Omega,C([a,b],X))} \to 0.$$

It follows that

$$\left\|\delta_t\bar{h} - \delta_t\bar{h}_n\right\|_{L_2(\Omega,X)} \to 0,$$

which together with (15) and (20) implies

$$\delta_t \bar{h} = \int_a^t T f(\tau) dW(\tau),$$

with equality considered in $L_2(\Omega,X)$. It follows that the function $\delta_t\bar{h}(\omega)$ is the desired continuous version, which proves the claim.

We define for $\omega \in \Omega$

$$S^T(f,\omega) = \bar{h}(\omega). \tag{22}$$

This way we obtain for each $T \in \mathscr{M}_2(Y,X)$ a mapping

$$S^T : L_2([a,b]\times\Omega, \mathscr{B}\mathscr{F}, Y) \times \Omega \to C([a,b],X) \tag{23}$$

such that for each $f \in L_2([a,b]\times\Omega, \mathscr{B}\mathscr{F}, Y)$ we have

$$S^T(f,\,\cdot\,) \in L_2(\Omega, C([a,b],X)) \tag{24}$$

and for each $t \in [a,b]$

$$\left(S^T(f,\,\omega)\right)(t) = \left(\int_a^t Tf(\tau)dW(\tau)\right)(\omega) \quad (\mathbb{P}\text{-a.s.}). \tag{25}$$

Using the Kolmogorov-Doob inequality once more, it follows that

$$\begin{aligned} &\left(\mathbb{E}\sup_{t\in[a,b]}\left\|\left(S^T(f,\,\omega)\right)(t)\right\|_X^2\right)^{1/2} \\ \le\ & 2\left(\mathbb{E}\left\|\int_a^b Tf(\tau)dW(\tau)\right\|_X^2\right)^{1/2} \le 2\mu_2(T)\left(\int_a^b \mathbb{E}\,\|f(\tau)\|_Y^2 d\tau\right)^{1/2}. \end{aligned} \tag{26}$$

We also consider the respective mapping for definite stochastic integration

$$S_1^T : L_2([a,b]\times\Omega,\mathscr{B}\mathscr{F},Y)\times\Omega \to X, \tag{27}$$

$$S_1^T(f,\omega) = (S^T(f,\omega))(b), \tag{28}$$

which has the analogous properties

$$S_1^T(f,\,\cdot\,) \in L_2(\Omega,X) \tag{29}$$

and

$$S_1^T(f,\,\omega) = \left(\int_a^b Tf(\tau)dW(\tau)\right)(\omega) \quad (\mathbb{P}\text{-a.s.}). \tag{30}$$

Finally we note the following. For $T,T_1,T_2 \in \mathscr{M}_2(Y,X)$, $f \in L_2([a,b]\times\Omega,\mathscr{B}\mathscr{F},Y)$, and $\gamma_1,\gamma_2 \in \mathbb{R}$ we have

$$S^{\gamma_1 T_1+\gamma_2 T_2}(f,\omega) = \gamma_1 S^{T_1}(f,\omega) + \gamma_2 S^{T_2}(f,\omega) \quad (\mathbb{P}\text{-a.s.}), \tag{31}$$

and for Banach spaces X_1,Y_1, operators $U \in \mathscr{L}(Y_1,Y)$, $V \in \mathscr{L}(X,X_1)$, $f \in L_2([a,b]\times\Omega,\mathscr{B}\mathscr{F},Y_1)$

$$S^{VTU}(f,\omega) = VS^T(Uf,\omega) \quad (\mathbb{P}\text{-a.s.}). \tag{32}$$

Indeed, let us verify (32), the proof of (31) is analogous. From (25) we conclude for each $t \in [a,b]$

$$\begin{aligned} \left(S^{VTU}(f,\omega)\right)(t) &= \left(\int_a^t VTUf(\tau)dW(\tau)\right)(\omega) \\ &= V\left(\int_a^t TUf(\tau)dW(\tau)\right)(\omega) = V\left(S^T(Uf,\omega)\right)(t) \quad (\mathbb{P}\text{-a.s.}). \end{aligned}$$

This, in turn, implies that $\mathbb{P}$-almost surely the following holds

$$\left(S^{VTU}(f,\omega)\right)(t) \;=\; V\left(S^T(Uf,\omega)\right)(t) \quad (t \in [a,b]\cap\mathbb{Q}), \tag{33}$$

where $\mathbb{Q}$ stands for the set of rationals. But then continuity implies that (33) holds for all $t \in [a,b]$, which yields (32).

The respective statements (31) and (32) also hold for S_1, which follows by setting $t = 1$.

3 Approximation of Banach space valued stochastic integrals

Let X, Y be Banach spaces, $T \in \mathscr{M}_2(Y, X)$, $0 \le \varrho \le 1$, and $\kappa > 0$. Let $F^\varrho([0,1] \times \Omega, Y; \kappa)$ be the set of all functions $f \in L_2([0,1] \times \Omega, \mathscr{B}\mathscr{F}, Y)$ such that

$$\left(\mathbb{E}\, \|f(0,\omega)\|_Y^2\right)^{1/2} \le \kappa, \tag{34}$$

$$\left(\mathbb{E}\, \|f(t,\omega) - f(t',\omega)\|_Y^2\right)^{1/2} \le \kappa |t - t'|^\varrho \quad (t, t' \in [0,1]). \tag{35}$$

Moreover, let $2 < q < \infty$ and let $F_q^\varrho([0,1] \times \Omega, Y; \kappa)$ be the subset of all $f \in F^\varrho([0,1] \times \Omega, Y; \kappa)$ such that for all finite subsets $M \subset [0,1]$

$$\left(\mathbb{E} \max_{t \in M} \|f(t,\omega)\|_Y^q\right)^{1/q} \le \kappa. \tag{36}$$

We want to approximate the operators of indefinite and definite stochastic integration (22–30), with $a = 0$, $b = 1$, which we consider here as acting on $F^\varrho([0,1] \times \Omega, Y; \kappa)$, thus

$$S^T : F^\varrho([0,1] \times \Omega, Y; \kappa) \times \Omega \to C([0,1], X), \tag{37}$$

$$S_1^T : F^\varrho([0,1] \times \Omega, Y; \kappa) \times \Omega \to X. \tag{38}$$

Let $n \in \mathbb{N}$ and $t_k = k/n$ $(k = 0, \dots, n)$. We set $z_0(\omega) = 0$, use the Euler-Maruyama scheme

$$z_{k+1}(\omega) = z_k(\omega) + f(t_k, \omega)(W(t_{k+1}, \omega) - W(t_k, \omega)) \quad (k = 0, \dots, n-1),$$

and define

$$A_n(f, \omega) = z \in C([0,1], Y),$$

where for $t \in [t_k, t_{k+1}]$, $0 \le k \le n-1$

$$\begin{aligned} z(t,\omega) &= z_k(\omega) + n(t - t_k)(z_{k+1}(\omega) - z_k(\omega)) \\ &= z_k(\omega) + n(t - t_k) f(t_k, \omega)(W(t_{k+1}, \omega) - W(t_k, \omega)). \end{aligned} \tag{39}$$

From (3) and (39) we obtain

$$\begin{aligned} A_n(f,\omega) &= P_n^{1,1,Y} (z_k(\omega))_{k=0}^n \\ &= P_n^{1,1,Y} \left(\sum_{j=0}^{k-1} f(t_j, \omega)(W(t_{j+1}, \omega) - W(t_j, \omega)) \right)_{k=0}^n \\ &= \sum_{k=0}^n \left(\sum_{j=0}^{k-1} f(t_j, \omega)(W(t_{j+1}, \omega) - W(t_j, \omega)) \right) \varphi_{n,t_k}^{1,1}, \end{aligned} \tag{40}$$

with $\varphi^{1,1}_{n,t_k}$ being the usual hat functions corresponding to the grid $\Gamma^1_n = \{t_k : 0 \le k \le n\}$, compare (2).

For the case of definite integration we set

$$A_{n,1}(f,\omega) \;=\; (A_n(f,\omega))(1) = \sum_{j=0}^{n-1} f(t_j,\omega)(W(t_{j+1},\omega) - W(t_j,\omega)). \quad (41)$$

Proposition 3.1. *Let $0 \le \varrho \le 1$, $2 < q < \infty$, $\kappa > 0$. Then there are constants $c_1, c_2 > 0$ such that for all X, Y, T as above and all $f \in F^\varrho([0,1]\times\Omega, Y;\kappa)$*

$$\left(\mathbb{E}\,\|S^T_1(f,\omega) - TA_{n,1}(f,\omega)\|^2_X\right)^{1/2} \;\le\; c_1\mu_2(T)n^{-\varrho}. \quad (42)$$

Moreover, if $f \in F^\varrho_q([0,1]\times\Omega, Y;\kappa)$, then

$$\begin{aligned}&\left(\mathbb{E}\,\|S^T(f,\omega) - TA_n(f,\omega)\|^2_{C([0,1],X)}\right)^{1/2}\\ &\quad\le\; c_2\mu_2(T)n^{-\varrho} + c_2\|T\|n^{-1/2}(\log n + 1)^{1/2}. \quad (43)\end{aligned}$$

Proof. Let $f \in F^\varrho([0,1]\times\Omega, Y;\kappa)$. We set for $t \in [0,1]$, $\omega \in \Omega$

$$u(t,\omega) = \left(S^T(f,\omega)\right)(t) \quad (44)$$
$$u_k(\omega) = Tz_k(\omega), \quad v(t,\omega) = Tz(t,\omega). \quad (45)$$

Using (16) and the Kolmogorov-Doob inequality, we get from (25), (31), and (35),

$$\begin{aligned}&\mathbb{E}\max_{0\le k\le n}\|u(t_k,\omega) - u_k(\omega)\|^2_X\\ &\quad=\; \mathbb{E}\max_{1\le k\le n}\left\|\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} T(f(t) - f(t_j))dW(t)\right\|^2_X\\ &\quad\le\; 4\mathbb{E}\left\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} T(f(t) - f(t_j))dW(t)\right\|^2_X\\ &\quad\le\; 4\mu_2(T)^2\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\mathbb{E}\,\|f(t) - f(t_j)\|^2_Y dt\\ &\quad\le\; 4\mu_2(T)^2 n^{-2\varrho}, \quad (46)\end{aligned}$$

proving, in particular, (42). Now we assume $f \in F^\varrho_q([0,1]\times\Omega, Y;\kappa)$ and show (43). Observe that by (39), (44), and (45), for $t \in [t_k, t_{k+1}]$

$$\begin{aligned}&u(t,\omega) - v(t,\omega)\\ &\quad=\; u(t_k,\omega) - u_k(\omega) + \left(S^T(f,\omega)\right)(t) - \left(S^T(f,\omega)\right)(t_k)\\ &\qquad -n(t-t_k)Tf(t_k,\omega)(W(t_{k+1},\omega) - W(t_k,\omega))\\ &\quad=\; u(t_k,\omega) - u_k(\omega) + \Phi_k(t,\omega) + Tf(t_k,\omega)(W(t,\omega) - \Psi(t,\omega)),\end{aligned}$$

where we defined for $t \in [t_k, t_{k+1}]$

$$\begin{aligned} \Phi_k(t,\omega) &= \left(S^T(f,\omega)\right)(t) - \left(S^T(f,\omega)\right)(t_k) \\ &\quad - f(t_k,\omega)(W(t,\omega) - W(t_k,\omega)) \end{aligned} \tag{47}$$

and for $t \in [0,1]$

$$\Psi(t,\omega) = W(t_k,\omega) + n(t-t_k)(W(t_{k+1},\omega) - W(t_k,\omega)). \tag{48}$$

The process $\Phi_k(t,\omega)$ is a continuous martingale on $[t_k, t_{k+1}]$, while $\Psi(t,\omega)$ is the piecewise linear interpolation of the scalar Wiener process. Consequently,

$$\begin{aligned} &\left(\mathbb{E} \sup_{0\le t\le 1} \|u(t,\omega) - v(t,\omega)\|_X^2\right)^{1/2} \\ &\le \left(\mathbb{E} \max_{0\le k\le n-1} \|u(t_k,\omega) - u_k(\omega)\|_X^2\right)^{1/2} \\ &+ \left(\mathbb{E} \max_{0\le k\le n-1} \sup_{t_k\le t\le t_{k+1}} \|\Phi_k(t,\omega)\|_X^2\right)^{1/2} \\ &\quad + \left(\mathbb{E} \max_{0\le k\le n-1} \sup_{t_k\le t\le t_{k+1}} \|Tf(t_k,\omega)(W(t,\omega) - \Psi(t,\omega))\|_X^2\right)^{1/2}. \end{aligned} \tag{49}$$

We have, using the Kolmogorov-Doob inequality, (16), (31), and (35),

$$\begin{aligned} &\left(\mathbb{E} \max_{0\le k\le n-1} \sup_{t_k\le t\le t_{k+1}} \|\Phi_k(t,\omega)\|_X^2\right)^{1/2} \\ &\le \left(\sum_{k=0}^{n-1} \mathbb{E} \sup_{t_k\le t\le t_{k+1}} \|\Phi_k(t,\omega)\|_X^2\right)^{1/2} \le 2\left(\sum_{k=0}^{n-1} \mathbb{E}\, \|\Phi_k(t_{k+1},\omega)\|_X^2\right)^{1/2} \\ &= 2\left(\sum_{k=0}^{n-1} \mathbb{E} \left\|\int_{t_k}^{t_{k+1}} T(f(\tau) - f(t_k))dW(\tau)\right\|_X^2\right)^{1/2} \\ &\le 2\mu_2(T)\left(\sum_{k=0}^{n-1} \mathbb{E} \int_{t_k}^{t_{k+1}} \|f(\tau) - f(t_k)\|_Y^2\, d\tau\right)^{1/2} \le 2\kappa\mu_2(T)n^{-\varrho}. \end{aligned} \tag{50}$$

We choose q_1 in such a way that $1/2 = 1/q + 1/q_1$ and use Hölder's inequality to conclude

$$\begin{aligned} &\left(\mathbb{E} \max_{0\le k\le n-1} \sup_{t_k\le t\le t_{k+1}} \|Tf(t_k,\omega)(W(t,\omega) - \Psi(t,\omega))\|_X^2\right)^{1/2} \\ &\le \left(\mathbb{E}\left(\max_{0\le k\le n-1} \|Tf(t_k,\omega)\|_X \sup_{0\le t\le 1} |W(t,\omega) - \Psi(t,\omega)|\right)^2\right)^{1/2} \\ &\le \left(\mathbb{E} \max_{0\le k\le n-1} \|Tf(t_k,\omega)\|_X^q\right)^{1/q} \left(\mathbb{E} \sup_{0\le t\le 1} |W(t,\omega) - \Psi(t,\omega)|^{q_1}\right)^{1/q_1}. \end{aligned} \tag{51}$$

It is well-known, see, e.g., [19], that the interpolation (48) of the scalar Wiener process satisfies

$$\left(\mathbb{E} \sup_{0\le t\le 1} |W(t,\omega)-\Psi(t,\omega)|^{q_1}\right)^{1/q_1} \quad\le\quad c_0 n^{-1/2}(\log n+1)^{1/2}. \tag{52}$$

It follows from (51–52) and (36) that

$$\begin{aligned}&\left(\mathbb{E} \max_{0\le k\le n-1} \sup_{t_k\le t\le t_{k+1}} \|Tf(t_k,\omega)(W(t,\omega)-\Psi(t,\omega))\|_X^2\right)^{1/2}\\ &\le\quad c_0\kappa\|T\|n^{-1/2}(\log n+1)^{1/2}.\end{aligned} \tag{53}$$

Combining (49), (46), (50), and (53), we get (43).

□

Remark. Clearly, the case of stochastic integration of functions with values in a space X of martingale type 2 is contained in the above, with $Y = X$ and $T = I_X$.

4 A general multilevel algorithm for Banach space valued stochastic integrals

Let X, Y be Banach spaces, $T, T_l \in \mathscr{M}_2(Y,X)$ ($l \in \mathbb{N}_0$), and let $l_1 \in \mathbb{N}_0$, $n_0,\dots,n_{l_1} \in \mathbb{N}$. For $f \in F^\varrho([0,1]\times\Omega, Y;\kappa)$, $\omega\in\Omega$ we define an approximation $\mathcal{A}(f,\omega)$ to $S(f,\omega)$ by

$$\mathcal{A}(f,\omega) = \sum_{l=0}^{l_1}(T_l - T_{l-1})A_{n_l}(f,\omega),$$

with the convention $T_{-1} = 0$, and analogously, an approximation $\mathcal{A}_1(f,\omega)$ to $S_1(f,\omega)$ in the definite case.

Proposition 4.1. *Let $0\le\varrho\le 1$, $2<q<\infty$, $\kappa>0$. Then there are constants $c_1, c_2 > 0$ such that for all $X, Y, T, (T_l)_{l=0}^\infty$ as above, $l_1\in\mathbb{N}_0$, $n_0,\dots,n_{l_1}\in\mathbb{N}$, $f\in F^\varrho([0,1]\times\Omega,Y;\kappa)$,*

$$\begin{aligned}&\big(\mathbb{E}\,\|S_1^T(f,\omega)-\mathcal{A}_1(f,\omega)\|_X^2\big)^{1/2}\\ &\le\quad c_1\mu_2(T-T_{l_1}) + c_1\sum_{l=0}^{l_1}\mu_2(T_l-T_{l-1})n_l^{-\varrho}\end{aligned} \tag{54}$$

and, if $f\in F_q^\varrho([0,1]\times\Omega,Y;\kappa)$,

$$\begin{aligned}&\big(\mathbb{E}\,\|S^T(f,\omega)-\mathcal{A}(f,\omega)\|_{C([0,1],X)}^2\big)^{1/2}\\ &\le\quad c_2\mu_2(T-T_{l_1})\\ &\quad +c_2\sum_{l=0}^{l_1}\left(\mu_2(T_l-T_{l-1})n_l^{-\varrho}+\|T_l-T_{l-1}\|n_l^{-1/2}(\log n_l+1)^{1/2}\right).\end{aligned} \tag{55}$$

Proof. We have, using (16), (25), (31–32), (34–35), and Proposition 3.1

$$\begin{aligned}
&\left(\mathbb{E}\,\|S_1^T(f,\omega)-\mathcal{A}_1(f,\omega)\|_X^2\right)^{1/2} \\
&\le \left(\mathbb{E}\,\|S_1^T(f,\omega)-S_1^{T_{l_1}}(f,\omega)\|_X^2\right)^{1/2} \\
&\quad + \sum_{l=0}^{l_1}\left(\mathbb{E}\,\|S_1^{T_l-T_{l-1}}(f,\omega)-(T_l-T_{l-1})A_{n_l,1}(f,\omega)\|_X^2\right)^{1/2} \\
&\le c\mu_2(T-T_{l_1})+\kappa\sum_{l=0}^{l_1}\mu_2(T_l-T_{l-1})n_l^{-\varrho}.
\end{aligned}$$

This shows (54), relation (55) follows analogously, using (26) instead of (16). □

In the following we estimate the type 2 constants of the operators involved above by the type 2 constants of their image spaces. We set for $l\in\mathbb{N}_0$

$$\begin{aligned}
X_l &= \mathrm{cl}_X(T_l(Y)) \\
X_{l,l-1} &= \mathrm{cl}_X((T_l-T_{l-1})(Y)),
\end{aligned}$$

where cl_X denotes the closure in the space X. Observe that $X_{0,-1}=X_0$. Clearly, we have

$$\mu_2(T_l-T_{l-1}) \le \mu_2(X_{l,l-1})\|T_l-T_{l-1}\|_{\mathscr{L}(Y,X)}. \tag{56}$$

The following lemma is a direct consequence of (56). We omit the elementary proof.

Lemma 4.2. *Let $T\in\mathscr{L}(Y,X)$ and assume that*

$$\lim_{m\to\infty}\|T-T_m\|_{\mathscr{L}(Y,X)}=0.$$

Then

$$\begin{aligned}
\mu_2(T) &\le \sum_{l=0}^{\infty}\mu_2(X_{l,l-1})\|T_l-T_{l-1}\|_{\mathscr{L}(Y,X)}, \\
\mu_2(T-T_{l_1}) &\le \sum_{l=l_1+1}^{\infty}\mu_2(X_{l,l-1})\|T_l-T_{l-1}\|_{\mathscr{L}(Y,X)}.
\end{aligned}$$

5 Parametric scalar stochastic integrals

Let $r,d\in\mathbb{N}$, $Q=[0,1]^d$, and let $\mathcal{L}_0^r(Q\times[0,1]\times\Omega)$ denote the set of all functions $f:Q\times[0,1]\times\Omega\to\mathbb{R}$ such that for each $s\in Q$, $f(s,t,\omega)$ is progressively measurable and

$$f(\,\cdot\,,t,\omega)\in C^r(Q)\quad((t,\omega)\in[0,1]\times\Omega).$$

For $f \in \mathcal{L}_0^r(Q \times [0,1] \times \Omega)$ we define a function $\bar{f} : [0,1] \times \Omega \to C^r(Q)$ by setting for $t \in [0,1]$, and $\omega \in \Omega$,

$$\bar{f}(t,\omega) = f(\,\cdot\,,t,\omega). \tag{57}$$

Lemma 5.1. *If $f \in \mathcal{L}_0^r(Q \times [0,1] \times \Omega)$, then $\bar{f}$ is progressively measurable.*

Proof. Let $\delta_{s_0}^\alpha \in C^r(Q)^*$ be defined for $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le r$ and $s_0 \in Q$ by

$$\delta_{s_0}^\alpha(g) = \frac{\partial^{|\alpha|} g}{\partial s^\alpha}(s_0).$$

Let $\tau \in [0,1]$. We have for $t \in [0,\tau]$, $\omega \in \Omega$

$$\delta_{s_0}^0\left(\bar{f}(t,\omega)\right) = f(s_0,t,\omega).$$

It follows from the properties of f that $\delta_{s_0}^0 \circ \bar{f}$, restricted to $[0,\tau] \times \Omega$ is $\mathscr{B}([0,\tau]) \times \mathscr{F}_\tau$ measurable. Now assume $|\alpha| = 1$, with say $\alpha = e_i$, the i-th unit vector, and let $(\beta_j)_{j=1}^\infty \subset \mathbb{R}$ be any sequence with $\beta_j \ne 0$, $\beta_j \to 0$ as $j \to \infty$ and $(s_0 + \beta_j e_i)_{j=1}^\infty \subset Q$. Then we have for $t \in [0,\tau]$, $\omega \in \Omega$

$$\delta_{s_0}^\alpha\left(\bar{f}(t,\omega)\right) = \lim_{j\to\infty} \frac{\delta_{s_0+\beta_j e_i}^0\left(\bar{f}(t,\omega)\right) - \delta_{s_0}^0\left(\bar{f}(t,\omega)\right)}{\beta_j},$$

hence $\delta_{s_0}^\alpha \circ \bar{f}$, restricted to $[0,\tau] \times \Omega$ is $\mathscr{B}([0,\tau]) \times \mathscr{F}_\tau$ measurable. Continuing this way, we obtain that the same holds for all α with $|\alpha| \le r$.

Let $g \in C^r(Q)$, $n \in \mathbb{N}$, $a > 0$, and let $(s_i)_{i=1}^\infty \subset Q$ be a dense sequence. Then

$$\begin{aligned} & \left\{(t,\omega) \in [0,\tau] \times \Omega : \ \bar{f}(t,\omega) \in g + aB_{C^r(Q)}\right\} \\ = & \bigcap_{i\in\mathbb{N}, |\alpha|\le r} \left\{(t,\omega) \in [0,\tau] \times \Omega : \ \delta_{s_i}^\alpha \circ \bar{f}(t,\omega) \in [\delta_{s_i}^\alpha(g) - a, \delta_{s_i}^\alpha(g) + a]\right\} \\ \in & \ \mathscr{B}([0,\tau]) \times \mathscr{F}_\tau, \end{aligned}$$

which shows the progressive measurability of $\bar{f}$.

$\square$

Now let $0 \le \varrho \le 1$, $2 < q < \infty$ and let $F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)$ denote the set of all functions $f \in \mathcal{L}_0^r(Q \times [0,1] \times \Omega)$ such that

$$\left(\mathbb{E}\, \|f(\,\cdot\,,0,\omega)\|_{C^r(Q)}^2\right)^{1/2} \ \le \ \kappa, \tag{58}$$

$$\left(\mathbb{E}\, \|f(\cdot\,,t,\omega) - f(\cdot\,,t',\omega)\|_{C^r(Q)}^2\right)^{1/2} \ \le \ \kappa|t-t'|^\varrho \quad (t,t' \in [0,1]). \tag{59}$$

Moreover, let $F_q^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)$ be the subset of those $f \in F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)$ which fulfill

$$\left(\mathbb{E} \max_{t\in M} \|f(\,\cdot\,,t,\omega)\|_{C^r(Q)}^q\right)^{1/q} \ \le \ \kappa \quad (M \subset [0,1], |M| < \infty). \tag{60}$$

Next we connect the parametric setting to the Banach space valued setting. We put

$$X = C(Q), \quad Y = C^r(Q), \quad T = J : C^r(Q) \to C(Q),$$

where J is the embedding map. For $f \in F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)$ let $\bar{f} : [0,1] \times \Omega \to C^r(Q)$ be given by (57). It follows from Lemma 5.1 and the assumptions (58–60) that $\bar{f} \in F^{\varrho}([0,1] \times \Omega, C^r(Q); \kappa)$, and if $f \in F_q^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)$, then $\bar{f} \in F_q^{\varrho}([0,1] \times \Omega, C^r(Q); \kappa)$. Next we set

$$T_l = P_{2^l}^{r,d} J. \tag{61}$$

Corresponding to the convention $T_{-1} = 0$ we define $P_{2^{-1}}^{r,d} = 0$. We have

$$X_{l-1} = P_{2^{l-1}}^{r,d}(C^r(Q)) \subseteq P_{2^l}^{r,d}(C^r(Q)) = X_l,$$

and therefore also $X_{l,l-1} \subseteq X_l$. Furthermore, it follows from (4) and the interpolation property that

$$\left\| P_{2^l}^{r,d} : \ell_\infty^{\dim X_l} \to X_l \right\| \le c_0, \qquad \left\| \left(P_{2^l}^{r,d} \right)^{-1} : X_l \to \ell_\infty^{\dim X_l} \right\| = 1.$$

Hence, using Lemma 2.1, we obtain for $l \ge 1$

$$\mu_2(X_{l,l-1}) \le \mu_2(X_l) \le c(l+1)^{1/2}.$$

Moreover, by (5), $\left\| J - P_{2^l}^{r,d} J \right\|_{\mathscr{L}(C^r(Q), C(Q))} \to 0$ as $l \to \infty$ and

$$\left\| P_{2^l}^{r,d} J - P_{2^{l-1}}^{r,d} J \right\|_{\mathscr{L}(C^r(Q), C(Q))} \le c 2^{-rl}. \tag{62}$$

Thus, Lemma 4.2 yields

$$\begin{aligned} \mu_2(J : C^r(Q) \to C(Q)) &\le \sum_{l=0}^{\infty} \mu_2(X_{l,l-1}) \left\| \left(P_{2^l}^{r,d} - P_{2^{l-1}}^{r,d} \right) J \right\|_{\mathscr{L}(C^r(Q), C(Q))} \\ &\le c \sum_{l=0}^{\infty} (l+1)^{1/2} 2^{-rl} < \infty \end{aligned} \tag{63}$$

and

$$\mu_2 \left(J - P_{2^{l_1}}^{r,d} J : C^r(Q) \to C(Q) \right) \le c(l_1+1)^{1/2} 2^{-rl_1}. \tag{64}$$

It follows that S^J and S_1^J are well-defined. We have

$$S^J(\bar{f}, \omega) \in C([0,1], C(Q)) = C(Q \times [0,1]) \tag{65}$$

(canonical identification) and

$$S_1^J(\bar{f}, \omega) \in C(Q).$$

Now we are ready to define the operators of indefinite and definite parametric stochastic integration

$$\begin{aligned} \mathscr{S} : F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa) \times \Omega &\to C(Q \times [0,1]) & (66)\\ \mathscr{S}_1 : F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa) \times \Omega &\to C(Q) & (67)\end{aligned}$$

by setting

$$\mathscr{S}(f,\omega) = S^J(\bar{f},\omega), \quad \mathscr{S}_1(f,\omega) = S_1^J(\bar{f},\omega). \tag{68}$$

It follows from (28), recalling also the canonical identification (65), that

$$\begin{aligned} (\mathscr{S}_1(f,\omega))(s) &= \left(S_1^J(\bar{f},\omega)\right)(s) = \left(\left(S^J(\bar{f},\omega)\right)(1)\right)(s) \\ &= \left(S^J(\bar{f},\omega)\right)(s,1) = (\mathscr{S}(f,\omega))(s,1) \quad (s \in Q, \omega \in \Omega). \end{aligned} \tag{69}$$

Moreover, we conclude from (11) and (25) that for $s \in Q$, $t \in [0,1]$

$$\begin{aligned} (\mathscr{S}(f,\,\cdot\,))(s,t) &= \left\langle (S^J(\bar{f},\,\cdot\,))(t), \delta_s \right\rangle = \left\langle \int_0^t J\bar{f}(\tau)dW(\tau), \delta_s \right\rangle \\ &= \int_0^t \left\langle J\bar{f}(\tau), \delta_s \right\rangle dW(\tau) = \int_0^t f(s,\tau)dW(\tau), \end{aligned} \tag{70}$$

with equality considered in $L_2(\Omega)$. For $t = 1$ and $s \in Q$ we get from (28) and (70)

$$(\mathscr{S}_1(f,\,\cdot\,))(s) = (\mathscr{S}(f,\,\cdot\,))(s,1) = \int_0^1 f(s,\tau)dW(\tau) \tag{71}$$

in $L_2(\Omega)$. Thus, we obtained continuous versions of the processes given by parametric indefinite and definite stochastic integration.

Now we introduce the corresponding version of the general multilevel scheme of Section 4. Fix $l_1 \in \mathbb{N}_0$, $n_0, \dots, n_{l_1} \in \mathbb{N}$, let $f \in F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)$ and $\omega \in \Omega$. For the indefinite problem we set

$$\mathscr{A}(f,\omega) = \sum_{l=0}^{l_1} \left(P_{2^l}^{r,d} - P_{2^{l-1}}^{r,d}\right) \left(A_{n_l}(f_s,\omega)\right)_{s \in \Gamma_{r2^l}^d}, \tag{72}$$

where f_s is given by $f_s(t,\omega) := f(s,t,\omega)$ ($t \in [0,1]$, $\omega \in \Omega$) and $P_{2^{-1}} := 0$. In the definite case we put

$$\mathscr{A}_1(f,\omega) = \sum_{l=0}^{l_1} \left(P_{2^l}^{r,d} - P_{2^{l-1}}^{r,d}\right) \left(A_{n_l,1}(f_s,\omega)\right)_{s \in \Gamma_{r2^l}^d}. \tag{73}$$

From (61) we obtain

$$\mathscr{A}(f,\omega) = \mathcal{A}(\bar{f},\omega), \quad \mathscr{A}_1(f,\omega) = \mathcal{A}_1(\bar{f},\omega). \tag{74}$$

Taking into account (40), we can rewrite (72) as follows. For $s \in Q$, $t \in [0,1]$

$$
\begin{aligned}
&(\mathscr{A}(f,\omega))(s,t)\\
&= \sum_{l=0}^{l_1}\sum_{k=0}^{n_l}\left(\sum_{\sigma\in\Gamma^d_{r2^l}} u_{lk}(\sigma,\omega)\varphi^{r,d}_{2^l,\sigma}(s) - \sum_{\sigma\in\Gamma^d_{r2^{l-1}}} u_{l-1,k}(\sigma,\omega)\varphi^{r,d}_{2^{l-1},\sigma}(s)\right)\varphi^{1,1}_{n_l,\frac{k}{n_l}}(t)
\end{aligned}
$$

with

$$
u_{lk}(\sigma,\omega) = \sum_{j=0}^{k-1} f\left(\sigma,\frac{j}{n_l},\omega\right)\left(W\left(\frac{j+1}{n_l},\omega\right) - W\left(\frac{j}{n_l},\omega\right)\right) \quad (0 \le l \le l_1)
$$

and $u_{lk} \equiv 0$ for $l = -1$. Similarly we obtain from (41) and (73), for $s \in Q$,

$$
\begin{aligned}
&(\mathscr{A}_1(f,\omega))(s)\\
&= \sum_{l=0}^{l_1}\left(\sum_{\sigma\in\Gamma^d_{r2^l}} u_{ln_l}(\sigma,\omega)\varphi^{r,d}_{2^l,\sigma}(s) - \sum_{\sigma\in\Gamma^d_{r2^{l-1}}} u_{l-1,n_l}(\sigma,\omega)\varphi^{r,d}_{2^{l-1},\sigma}(s)\right).
\end{aligned}
$$

Let $\mathrm{card}(A)$, $A = \mathscr{A}_1, \mathscr{A}$, denote the number of values of f and W used in algorithm A (see Section 6 for a general definition in the context of complexity theory). Then we have

$$
\mathrm{card}(\mathscr{A}) = \mathrm{card}(\mathscr{A}_1) \le c\sum_{l=0}^{l_1} n_l 2^{dl}. \tag{75}
$$

Proposition 5.2. *Let $r,d \in \mathbb{N}$, $0 \le \varrho \le 1$, $2 < q < \infty$, $\kappa > 0$. Then there are constants $c_1, c_2 > 0$ such that for all $l_1 \in \mathbb{N}_0$, $n_0,\dots,n_{l_1} \in \mathbb{N}$, $f \in F^{r,\varrho}(Q \times [0,1] \times \Omega;\kappa)$,*

$$
\begin{aligned}
&\left(\mathbb{E}\,\|\mathscr{S}_1(f,\omega) - \mathscr{A}_1(f,\omega)\|^2_{C(Q)}\right)^{1/2}\\
&\le c_1(l_1+1)^{1/2}2^{-rl_1} + c_1\sum_{l=0}^{l_1}(l+1)^{1/2}2^{-rl}n_l^{-\varrho}
\end{aligned} \tag{76}
$$

and for all $f \in F^{r,\varrho}_q(Q \times [0,1] \times \Omega;\kappa)$

$$
\begin{aligned}
&\left(\mathbb{E}\,\|\mathscr{S}(f,\omega) - \mathscr{A}(f,\omega)\|^2_{C(Q\times[0,1])}\right)^{1/2}\\
&\le c_2(l_1+1)^{1/2}2^{-rl_1} + c_2\sum_{l=0}^{l_1}2^{-rl}\left((l+1)^{1/2}n_l^{-\varrho} + n_l^{-1/2}(\log n_l + 1)^{1/2}\right).
\end{aligned} \tag{77}
$$

Proof. The result follows from Proposition 4.1 together with (61–64), (68), and (74). □

Theorem 5.3. *Let $r, d \in \mathbb{N}$, $0 \le \varrho \le 1$, $2 < q < \infty$. There are constants $c_{1-4} > 0$ such that the following hold. For each $n \in \mathbb{N}$ with $n \ge 2$ there is a choice of $l_1 \in \mathbb{N}_0$ and $n_0, \dots, n_{l_1} \in \mathbb{N}_0$ such that* $\operatorname{card}(\mathscr{A}_1) \le c_1 n$ *and*

$$\sup_{f \in F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)} (\mathbb{E}\, \|\mathscr{S}_1(f,\omega) - \mathscr{A}_1(f,\omega)\|^2_{C(Q)})^{1/2} \le c_2 \begin{cases} n^{-r/d}(\log n)^{1/2} & \text{if } r/d < \varrho, \\ n^{-r/d}(\log n)^{r/d+3/2} & \text{if } r/d = \varrho, \\ n^{-\varrho} & \text{if } r/d > \varrho. \end{cases} \tag{78}$$

Similarly, for each $n \in \mathbb{N}$ with $n \ge 2$ there are $l_1 \in \mathbb{N}_0$ and $n_0, \dots, n_{l_1} \in \mathbb{N}_0$ such that $\operatorname{card}(\mathscr{A}) \le c_3 n$ *and*

$$\sup_{f \in F_q^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)} (\mathbb{E}\, \|\mathscr{S}(f,\omega) - \mathscr{A}(f,\omega)\|^2_{C(Q \times [0,1])})^{1/2} \le c_4 \begin{cases} n^{-r/d}(\log n)^{1/2} & \text{if } r/d < \min(\varrho, 1/2) \\ n^{-r/d}(\log n)^{r/d+3/2} & \text{if } r/d = \min(\varrho, 1/2), \\ n^{-1/2}(\log n)^{1/2} & \text{if } r/d > \min(\varrho, 1/2),\ \varrho \ge 1/2, \\ n^{-\varrho} & \text{if } r/d > \min(\varrho, 1/2),\ \varrho < 1/2. \end{cases} \tag{79}$$

Proof. Let $n \in \mathbb{N}, n \ge 2$, and put

$$l_1 = \left\lceil \frac{\log n}{d} \right\rceil. \tag{80}$$

Furthermore, let $\theta \in \{0,1\}, \delta_0, \delta_1 \ge 0$, and set for $l \in \mathbb{N}_0$, $l \le l_1$

$$n_l = \left\lceil (l_1+1)^{-\theta} 2^{d(l_1-l)-\delta_0 l - \delta_1(l_1-l)} \right\rceil. \tag{81}$$

We start with definite integration. From (81) we obtain

$$\begin{aligned} (l+1)^{1/2} 2^{-rl} n_l^{-\varrho} &\le (l+1)^{1/2}(l_1+1)^{\theta\varrho} 2^{-rl-\varrho d(l_1-l)+\varrho\delta_0 l+\varrho\delta_1(l_1-l)} \\ &= (l_1+1)^{\theta\varrho}(l+1)^{1/2} 2^{-(r-\varrho\delta_0)l-\varrho(d-\delta_1)(l_1-l)}. \end{aligned} \tag{82}$$

Combining (76) and (82), we conclude

$$\begin{aligned} E_1 &:= \sup_{f \in F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)} \left(\mathbb{E}\, \|\mathscr{S}_1(f,\omega) - \mathscr{A}_1(f,\omega)\|^2_{C(Q)}\right)^{1/2} \\ &\le c(l_1+1)^{1/2} 2^{-rl_1} + c(l_1+1)^{\theta\varrho} \sum_{l=0}^{l_1} (l+1)^{1/2} 2^{-(r-\varrho\delta_0)l-\varrho(d-\delta_1)(l_1-l)}. \end{aligned} \tag{83}$$

First we consider the case $r/d < \varrho$. Here we set $\theta = \delta_0 = 0$ and choose $\delta_1 > 0$ in such a way that $\varrho(d - \delta_1) > r$. Using (83), we obtain

$$\begin{aligned} E_1 &\le c(l_1+1)^{1/2} 2^{-rl_1} + c \sum_{l=0}^{l_1} (l+1)^{1/2} 2^{-rl-\varrho(d-\delta_1)(l_1-l)} \\ &\le c(l_1+1)^{1/2} 2^{-rl_1} \le cn^{-r/d}(\log n)^{1/2}. \end{aligned}$$

If $r/d = \varrho$, we let $\theta = 1$, $\delta_0 = \delta_1 = 0$. Then (83) yields

$$\begin{aligned} E_1 &\le c(l_1+1)^{1/2}2^{-rl_1} + c(l_1+1)^{r/d}\sum_{l=0}^{l_1}(l+1)^{1/2}2^{-rl_1} \\ &\le c(l_1+1)^{r/d+3/2}2^{-rl_1} \le cn^{-r/d}(\log n)^{r/d+3/2}. \end{aligned}$$

Finally, if $r/d > \varrho$, we set $\theta = \delta_1 = 0$ and choose $\delta_0 > 0$ such that $r - \varrho\delta_0 > \varrho d$. From (83) we obtain

$$\begin{aligned} E_1 &\le c(l_1+1)^{1/2}2^{-rl_1} + \sum_{l=0}^{l_1}(l+1)^{1/2}2^{-(r-\varrho\delta_0)l-\varrho d(l_1-l)} \\ &\le c\,2^{-\varrho d l_1} \le cn^{-\varrho}. \end{aligned}$$

Now we pass to indefinite integration. Using (81), we get

$$\begin{aligned} &2^{-rl}\left((l+1)^{1/2}n_l^{-\varrho} + n_l^{-1/2}(\log n_l+1)^{1/2}\right) \\ &\le c\,2^{-rl}\Big((l+1)^{1/2}(l_1+1)^{\theta\varrho}2^{-\varrho d(l_1-l)+\varrho\delta_0 l+\varrho\delta_1(l_1-l)} \\ &\qquad +(l_1-l+1)^{1/2}(l_1+1)^{\theta/2}2^{-d(l_1-l)/2+\delta_0 l/2+\delta_1(l_1-l)/2}\Big) \\ &= c(l_1+1)^{\theta\varrho}(l+1)^{1/2}2^{-(r-\varrho\delta_0)l-\varrho(d-\delta_1)(l_1-l)} \\ &\qquad +c(l_1+1)^{\theta/2}(l_1-l+1)^{1/2}2^{-(r-\delta_0/2)l-(d-\delta_1)(l_1-l)/2}. \end{aligned} \tag{84}$$

If $r/d < \min(\varrho, 1/2)$, we set $\theta = \delta_0 = 0$ and choose $\delta_1 > 0$ in such a way that $r < (d-\delta_1)\min(\varrho, 1/2)$. Using (77) and (84), we obtain

$$\begin{aligned} E &\le c(l_1+1)^{1/2}2^{-rl_1} + c\sum_{l=0}^{l_1}(l+1)^{1/2}2^{-rl-\varrho(d-\delta_1)(l_1-l)} \\ &\qquad +c\sum_{l=0}^{l_1}(l_1-l+1)^{1/2}2^{-rl-\frac{1}{2}(d-\delta_1)(l_1-l)} \\ &\le c(l_1+1)^{1/2}2^{-rl_1} \le cn^{-r/d}(\log n)^{1/2}. \end{aligned}$$

If $r/d = \min(\varrho, 1/2)$, we let $\theta = 1$, $\delta_0 = \delta_1 = 0$. Then (77) and (84) yield

$$\begin{aligned} E &\le c(l_1+1)^{1/2}2^{-rl_1} + c(l_1+1)^{\varrho}\sum_{l=0}^{l_1}(l+1)^{1/2}2^{-rl-\varrho d(l_1-l)} \\ &\qquad +c(l_1+1)^{1/2}\sum_{l=0}^{l_1}(l_1-l+1)^{1/2}2^{-rl-\frac{1}{2}d(l_1-l)}. \end{aligned} \tag{85}$$

We distinguish between three subcases. If $r/d = \varrho < 1/2$, then (85) yields

$$\begin{aligned} E &\leq c(l_1+1)^{1/2}2^{-rl_1} + c(l_1+1)^{\varrho+3/2}2^{-rl_1} + c(l_1+1)^{1/2}2^{-rl_1} \\ &\leq c(l_1+1)^{r/d+3/2}2^{-rl_1} \leq cn^{-r/d}(\log n)^{r/d+3/2}. \end{aligned}$$

If $r/d = \varrho = 1/2$, (85) gives

$$\begin{aligned} E &\leq c(l_1+1)^{1/2}2^{-rl_1} + c(l_1+1)^{\varrho+3/2}2^{-rl_1} + c(l_1+1)^{2}2^{-rl_1} \\ &\leq c(l_1+1)^{r/d+3/2}2^{-rl_1} \leq cn^{-r/d}(\log n)^{r/d+3/2}. \end{aligned}$$

Finally, if $r/d = 1/2 < \varrho$, we conclude from (85), taking into account that $\varrho \leq 1$,

$$\begin{aligned} E &\leq c(l_1+1)^{1/2}2^{-rl_1} + c(l_1+1)^{\varrho+1/2}2^{-rl_1} + c(l_1+1)^{2}2^{-rl_1} \\ &\leq c(l_1+1)^{r/d+3/2}2^{-rl_1} \leq cn^{-r/d}(\log n)^{r/d+3/2}. \end{aligned}$$

Now we assume $r/d > \min(\varrho, 1/2)$. Here we set $\theta = \delta_1 = 0$, and choose $\delta_0 > 0$ such that $r - \delta_0 \max(\varrho, 1/2) > d \min(\varrho, 1/2)$. Then (77) and (84) imply

$$\begin{aligned} E &\leq c(l_1+1)^{1/2}2^{-rl_1} + c\sum_{l=0}^{l_1}(l+1)^{1/2}2^{-(r-\varrho\delta_0)l-\varrho d(l_1-l)} \\ &\quad + c\sum_{l=0}^{l_1}(l_1-l+1)^{1/2}2^{-(r-\delta_0/2)l-\frac{1}{2}d(l_1-l)}. \end{aligned} \tag{86}$$

We consider two subcases. For $\varrho \geq 1/2$ we get from (86)

$$\begin{aligned} E &\leq c(l_1+1)^{1/2}2^{-rl_1} + c(l_1+1)^{1/2}\sum_{l=0}^{l_1}2^{-(r-\varrho\delta_0)l-\frac{1}{2}d(l_1-l)} \\ &\leq c(l_1+1)^{1/2}2^{-rl_1} + c(l_1+1)^{1/2}\ 2^{-\frac{1}{2}dl_1} \leq c\,n^{-1/2}(\log n)^{1/2}. \end{aligned}$$

If $\varrho < 1/2$, we have

$$(l_1-l+1)^{1/2}2^{-(r-\delta_0/2)l-\frac{1}{2}d(l_1-l)} \leq c\,2^{-(r-\delta_0/2)l-\varrho d(l_1-l)},$$

so we obtain from (86)

$$\begin{aligned} E &\leq c(l_1+1)^{1/2}2^{-rl_1} + c\sum_{l=0}^{l_1}(l+1)^{1/2}2^{-(r-\delta_0/2)l-\varrho d(l_1-l)} \\ &\leq c\,2^{-\varrho dl_1} \leq cn^{-\varrho}. \end{aligned}$$

This completes the proof of (78) and (79).

It follows from (75), (80), and (81) that

$$\begin{aligned} \operatorname{card}(\mathscr{A}_1) = \operatorname{card}(\mathscr{A}) &\leq c2^{dl_1} + c(l_1+1)^{-\theta}\sum_{l=1}^{l_1}2^{dl_1-\delta_0 l-\delta_1(l_1-l)} \\ &\leq c2^{dl_1} \leq cn, \end{aligned}$$

provided $\delta_0 + \delta_1 + \theta > 0$, which holds in all cases considered above.

$\square$

6 Complexity

We work in the framework of information-based complexity theory (IBC) as discussed in [20, 12]. Here we briefly describe the general setting for strong approximation of stochastic problems. An abstract problem is given by a tuple

$$\mathscr{P} = (F, (\Omega, \Sigma, \mathbb{P}), G, S, K, \Lambda).$$

Here F is a non-empty set, $(\Omega, \Sigma, \mathbb{P})$ a probability space, G a normed linear space, and S a mapping from $F \times \Omega$ to G, called solution operator. We assume that for each $f \in F$ the mapping $\omega \to S(f, \omega)$ is Σ-to-Borel-measurable and $\mathbb{P}$-almost surely separably valued, the latter meaning that for each $f \in F$ there is a separable subspace G_f of G such that

$$\mathbb{P}\{\omega : S(f, \omega) \in G_f\} = 0.$$

Furthermore, K is a non-empty set, and Λ is a set of mappings from $F \times \Omega$ to K, the set of admissible information functionals.

In this paper we restrict our attention to nonadaptive deterministic algorithms. An nonadaptive deterministic algorithm A for $\mathscr{P}$ is a tuple

$$A = (\lambda_1, \ldots, \lambda_n, \varphi),$$

where $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \Lambda$, and φ is an arbitrary mapping from K^n to G. The output $A(f, \omega) \in G$ of algorithm A at input $(f, \omega) \in F \times \Omega$ is defined as

$$A(f, \omega) = \varphi(\lambda_1(f, \omega), \ldots, \lambda_n(f, \omega)).$$

We require that for each $f \in F$ the mapping $\omega \to A(f, \omega)$ is Σ-to-Borel measurable and $\mathbb{P}$-almost surely separably valued. Note that the algorithm definition does not involve S. For a fixed $n \in \mathbb{N}$ we denote the set of all such algorithms by $\mathscr{A}_n^{\det}(F \times \Omega, G)$. The number n is called the cardinality of A. The error of an algorithm $A \in \mathscr{A}_n^{\det}(F \times \Omega, G)$ is defined by

$$e(S, A, F \times \Omega, G) = \sup_{f \in F} \mathbb{E} \, \|S(f, \omega) - A(f, \omega)\|_G.$$

Finally, the n-th minimal error is defined for $n \in \mathbb{N}$ as

$$e_n(S, F \times \Omega, G) = \inf_{A \in \mathscr{A}_n^{\det}(F \times \Omega, G)} e(S, A, F \times \Omega, G),$$

that is, $e_n(S, F \times \Omega, G)$ is the minimal possible error among all deterministic algorithms that use at most n information functionals.

6.1 Banach space valued setting

Let X, Y be Banach spaces, $T \in \mathscr{M}_2(Y, X)$, $T \neq 0$, $0 \leq \varrho \leq 1$, $\kappa > 0$, $2 < q < \infty$. For the definite integration problem we choose $F = F^\varrho([0,1] \times \Omega, Y; \kappa)$, $G = X$, $S = S_1^T$, $K = X \cup \mathbb{R}$, and

$$\Lambda^T = \{\delta_t^i : t \in [0,1], i = 0, 1\},$$

where for $f \in F$, $\omega \in \Omega$,

$$\delta_t^0(f, \omega) = Tf(t, \omega), \quad \delta_t^1(f, \omega) = W(t, \omega).$$

Note that a part of the information is Banach space valued. Thus the Banach space valued definite stochastic integration problem is given by

$$\mathcal{P}_1^T = (F^\varrho([0,1] \times \Omega, Y; \kappa), (\Omega, \Sigma, \mathbb{P}), X, S_1^T, X \cup \mathbb{R}, \Lambda^T).$$

In the indefinite case we choose $F = F_q^\varrho([0,1] \times \Omega, Y; \kappa)$, $G = C([0,1], X)$, $S = S^T$, $K = X \cup \mathbb{R}$, and $\Lambda = \Lambda^T$ as above. Then the indefinite problem is described by

$$\mathcal{P}^T = (F_q^\varrho([0,1] \times \Omega, Y; \kappa), (\Omega, \Sigma, \mathbb{P}), C([0,1], X), S^T, X \cup \mathbb{R}, \Lambda^T).$$

Next we state a complexity result for Banach space valued stochastic integration. We use the following abbreviations

$$\begin{aligned} e_n(S_1^T) &:= e_n(S_1^T, F^\varrho([0,1] \times \Omega, Y; \kappa) \times \Omega, X), \\ e_n(S^T) &:= e_n(S^T, F_q^\varrho([0,1] \times \Omega, Y; \kappa) \times \Omega, C([0,1], X)). \end{aligned}$$

Theorem 6.1. *Let X, Y be Banach spaces, $T \in \mathscr{M}_2(Y, X)$, $T \neq 0$, $\kappa > 0$, $0 \leq \varrho \leq 1$, $2 < q < \infty$. Then the following hold*

$$e_n(S_1^T) \asymp n^{-\varrho}$$

$$e_n(S^T) \asymp \begin{cases} n^{-\varrho} & \text{if} \quad \varrho < 1/2 \\ n^{-1/2}(\log n)^{1/2} & \text{if} \quad \varrho \geq 1/2. \end{cases}$$

Proof. The upper bounds follow immediately from Proposition 3.1. The lower bounds can be shown by reduction to the scalar valued setting: Let $y_0 \in Y$ be such that $\|y_0\|_Y = 1$ and $Ty_0 \neq 0$. Choose a $V \in X^*$ with $\|V\|_{X^*} = 1$ and $\langle Ty_0, V \rangle \neq 0$. Define the mapping $U : \mathbb{R} \to Y$ by $U\beta = \kappa\beta y_0$. Then $f \in B_{\mathscr{C}^\varrho([0,1])}$ implies $Uf \in F_q^\varrho([0,1] \times \Omega, Y; \kappa)$. Consider the scalar problems

$$\begin{aligned} \mathcal{P}_1^{I_\mathbb{R}} &= (B_{\mathscr{C}^\varrho([0,1])}, (\Omega, \Sigma, \mathbb{P}), \mathbb{R}, S_1^{I_\mathbb{R}}, \mathbb{R}, \Lambda^{I_\mathbb{R}}) \\ \mathcal{P}^{I_\mathbb{R}} &= (B_{\mathscr{C}^\varrho([0,1])}, (\Omega, \Sigma, \mathbb{P}), C([0,1]), S^{I_\mathbb{R}}, \mathbb{R}, \Lambda^{I_\mathbb{R}}), \end{aligned}$$

Let $f \in B_{\mathscr{C}^\varrho([0,1])}$. Since $VTU = \kappa \langle Ty_0, V \rangle$ (it is convenient for us to consider V also as an operator $V \in \mathscr{L}(X, \mathbb{R})$),we conclude from (31) and (32),

$$VS^T(Uf, \omega) = S^{VTU}(f, \omega) = \kappa \langle Ty_0, V \rangle S^{I_\mathbb{R}}(f, \omega) \quad (\mathbb{P}\text{-a.s.}). \tag{87}$$

and similarly,

$$VS_1^T(Uf,\omega) \;=\; \kappa\,\langle Ty_0, V\rangle\, S_1^{I_{\mathbb{R}}}(f,\omega) \quad (\mathbb{P}\text{-a.s.}), \tag{88}$$

It is easily checked that an algorithm

$$A \in \mathscr{A}_n^{\det}(F_q^\varrho([0,1]\times\Omega, Y;\kappa)\times\Omega, C([0,1],X))$$

for $\mathcal{P}^T$ induces an algorithm $\tilde{A} \in \mathscr{A}_n^{\det}(B_{\mathscr{C}^\varrho([0,1])}\times\Omega, C([0,1]))$ for $\mathcal{P}^{I_{\mathbb{R}}}$ given by

$$\tilde{A}(f,\omega) = \kappa^{-1}\,\langle Ty_0, V\rangle^{-1}\,\langle A(Uf,\omega), V\rangle\,.$$

Using (87), we conclude

$$\begin{aligned}
&e(S^{I_{\mathbb{R}}}, \tilde{A}, B_{\mathscr{C}^\varrho([0,1])}\times\Omega, C([0,1]))\\
&\quad= \sup_{f\in B_{\mathscr{C}^\varrho([0,1])}} \mathbb{E}\left\|S^{I_{\mathbb{R}}}(f,\omega)-\tilde{A}(f,\omega)\right\|_{C([0,1])}\\
&\quad= \kappa^{-1}\,|\langle Ty_0,V\rangle|^{-1} \sup_{f\in B_{\mathscr{C}^\varrho([0,1])}} \mathbb{E}\left\|\,\left\langle S^T(Uf,\omega),V\right\rangle - \langle A(Uf,\omega),V\rangle\,\right\|_{C([0,1])}\\
&\quad\le \kappa^{-1}\,|\langle Ty_0,V\rangle|^{-1} \sup_{f\in B_{\mathscr{C}^\varrho([0,1])}} \mathbb{E}\left\|S^T(Uf,\omega)-A(Uf,\omega)\right\|_{C([0,1],X)}\\
&\quad\le \kappa^{-1}\,|\langle Ty_0,V\rangle|^{-1}\, e(S^T, A, F_q^\varrho([0,1]\times\Omega,Y;\kappa)\times\Omega, C([0,1],X)),
\end{aligned}$$

hence

$$\begin{aligned}
&e_n(S^T, F_q^\varrho([0,1]\times\Omega,Y;\kappa)\times\Omega, C([0,1],X))\\
&\quad\ge\; \kappa\,|\langle Ty_0,V\rangle|\, e_n(S^{I_{\mathbb{R}}}, B_{\mathscr{C}^\varrho([0,1])}\times\Omega, C([0,1])).
\end{aligned}\tag{89}$$

Arguing similarly, we obtain from (88)

$$\begin{aligned}
&e_n(S_1^T, F^\varrho([0,1]\times\Omega,Y;\kappa)\times\Omega, X)\\
&\quad\ge\; \kappa\,|\langle Ty_0,V\rangle|\, e_n(S_1^{I_{\mathbb{R}}}, B_{\mathscr{C}^\varrho([0,1])}\times\Omega, \mathbb{R}).
\end{aligned}\tag{90}$$

It was shown in [18] that

$$e_n(S_1^{I_{\mathbb{R}}}, B_{\mathscr{C}^\varrho([0,1])}\times\Omega,\mathbb{R}) \ge cn^{-\varrho}. \tag{91}$$

Using this and (89), we derive

$$\begin{aligned}
&e_n(S^T, F_q^\varrho([0,1]\times\Omega,Y;\kappa)\times\Omega, C([0,1],X))\\
&\quad\ge\; \kappa\,|\langle Ty_0,V\rangle|\;\; e_n(S^{I_{\mathbb{R}}}, B_{\mathscr{C}^\varrho([0,1])}\times\Omega, C([0,1]))\\
&\quad\ge\; \kappa\,|\langle Ty_0,V\rangle|\;\; e_n(S_1^{I_{\mathbb{R}}}, B_{\mathscr{C}^\varrho([0,1])}\times\Omega, \mathbb{R}) \ge cn^{-\varrho},
\end{aligned}\tag{92}$$

where we used (28) again. Thus, the lower bound of the definite case in Theorem 6.1 follows from (90) and (91). For $\varrho < 1/2$ the lower bound for the indefinite case is a consequence of (92). Furthermore, the indefinite scalar case contains for $f \equiv 1$ the problem of approximation of the Wiener process $W(t,\omega)$ itself. Therefore we get from [19] and [11]

$$e_n(S^{I_{\mathbb{R}}}, B_{\mathscr{C}^{\varrho}([0,1])} \times \Omega, C([0,1])) \geq cn^{-1/2}(\log n)^{1/2}. \tag{93}$$

Thus, for $\varrho \geq 1/2$ the lower bound of the indefinite case follows from (92) and (93).

□

6.2 Parametric setting

For definite integration we choose $F = F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)$, $G = C(Q)$, $S = \mathscr{S}_1$, $K = \mathbb{R}$, and Λ is given by

$$\Lambda = \{\delta_{st} : s \in Q, t \in [0,1]\} \cup \{\delta_t : t \in [0,1]\},$$

where for $f \in F$, $\omega \in \Omega$,

$$\delta_{st}(f,\omega) = f(s,t), \quad \delta_t(f,\omega) = W(t,\omega).$$

Here we have $\mathbb{R}$ valued information consisting of values of f and W. So the definite integration problem is defined by

$$\mathscr{P}_1 = (F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa), (\Omega, \Sigma, \mathbb{P}), C(Q), \mathscr{S}_1, \mathbb{R}, \Lambda).$$

Moreover, for the indefinite problem we set $F = F_q^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa)$, $G = C(Q \times [0,1])$, $S = \mathscr{S}$, $K = \mathbb{R}$, and Λ as above, thus, the indefinite integration problem is described by

$$\mathscr{P} = (F_q^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa), (\Omega, \Sigma, \mathbb{P}), C(Q \times [0,1]), \mathscr{S}, \mathbb{R}, \Lambda).$$

We write for brevity

$$\begin{aligned} e_n(\mathscr{S}_1) &:= e_n(\mathscr{S}_1, F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa) \times \Omega, C(Q)), \\ e_n(\mathscr{S}) &:= e_n(\mathscr{S}, F_q^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa) \times \Omega, C(Q \times [0,1])). \end{aligned}$$

Theorem 6.2. *Let $r, d \in \mathbb{N}$, $\kappa > 0$, $0 \leq \varrho \leq 1$, and $2 < q < \infty$. Then we have in the definite case*

$$\begin{aligned} n^{-\frac{r}{d}} \preceq e_n(\mathscr{S}_1) &\preceq n^{-\frac{r}{d}}(\log n)^{\frac{1}{2}} && \text{if } \tfrac{r}{d} < \varrho \\ n^{-\frac{r}{d}} \preceq e_n(\mathscr{S}_1) &\preceq n^{-\frac{r}{d}}(\log n)^{\frac{r}{d}+\frac{3}{2}} && \text{if } \tfrac{r}{d} = \varrho \\ e_n(\mathscr{S}_1) &\asymp n^{-\varrho} && \text{if } \tfrac{r}{d} > \varrho. \end{aligned}$$

Moreover, in the indefinite case,

$$
\begin{aligned}
n^{-\frac{r}{d}} \preceq e_n(\mathscr{S}) &\preceq n^{-\frac{r}{d}}(\log n)^{\frac{1}{2}} && \text{if} \quad \varrho < \tfrac{1}{2} \wedge \tfrac{r}{d} < \varrho \\
n^{-\frac{r}{d}} \preceq e_n(\mathscr{S}) &\preceq n^{-\frac{r}{d}}(\log n)^{\frac{r}{d}+\frac{3}{2}} && \text{if} \quad \varrho < \tfrac{1}{2} \wedge \tfrac{r}{d} = \varrho \\
e_n(\mathscr{S}) &\asymp n^{-\varrho} && \text{if} \quad \varrho < \tfrac{1}{2} \wedge \tfrac{r}{d} > \varrho \\
n^{-\frac{r}{d}} \preceq e_n(\mathscr{S}) &\preceq n^{-\frac{r}{d}}(\log n)^{\frac{1}{2}} && \text{if} \quad \varrho \ge \tfrac{1}{2} \wedge \tfrac{r}{d} < \tfrac{1}{2} \\
n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} \preceq e_n(\mathscr{S}) &\preceq n^{-\frac{1}{2}}(\log n)^{2} && \text{if} \quad \varrho \ge \tfrac{1}{2} \wedge \tfrac{r}{d} = \tfrac{1}{2} \\
e_n(\mathscr{S}) &\asymp n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} && \text{if} \quad \varrho \ge \tfrac{1}{2} \wedge \tfrac{r}{d} > \tfrac{1}{2}.
\end{aligned}
$$

Proof. The upper bounds follow immediately from Theorem 5.3. To prove the lower bounds, define

$$R_1 : B_{C^r(Q)} \to F_q^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa), \quad (R_1(g))(s,t,\omega) := \kappa g(s).$$

Let $g \in B_{C^r(Q)}$. Then according to definition (57), $\overline{R_1(g)}(t,\omega) = \kappa g$. Using (68) and (30), we obtain

$$\mathscr{S}_1(R_1(g),\omega) = S_1^J\left(\overline{R_1(g)},\omega\right) = \kappa g\left(\int_0^1 dW(t)\right)(\omega) = \kappa W(1,\omega)g \quad (\mathbb{P}\text{-a.s.}).$$

Now let $A \in \mathscr{A}_n^{\det}(F^{r,\varrho}(Q \times [0,1] \times \Omega; \kappa) \times \Omega, C(Q))$. We have

$$
\begin{aligned}
& e(\mathscr{S}_1, A, R_1(B_{C^r(Q)}) \times \Omega, C(Q)) \\
&= \sup_{g \in B_{C^r(Q)}} \mathbb{E}\, \|\mathscr{S}_1(R_1(g),\omega) - A(R_1(g),\omega)\|_{C(Q)} \\
&= \sup_{g \in B_{C^r(Q)}} \mathbb{E}\, \|\kappa W(1,\omega)g - A(R_1(g),\omega)\|_{C(Q)}. \\
&\ge \mathbb{P}\{\omega : W(1,\omega) \ge 1\} \times \\
&\qquad \sup_{g \in B_{C^r(Q)}} \mathbb{E}\left(\|\kappa W(1,\omega)g - A(R_1(g),\omega)\|_{C(Q)} \big| W(1,\omega) \ge 1\right) \\
&\ge c \sup_{g \in B_{C^r(Q)}} \mathbb{E}\left(\left\| g - \frac{1}{\kappa W(1,\omega)} A(R_1(g),\omega)\right\|_{C(Q)} \middle| W(1,\omega) \ge 1\right). \qquad (94)
\end{aligned}
$$

Set $\Omega_1 := \{\omega \in \Omega : W(1,\omega) \ge 1\}$ and define

$$A_\omega^1(g) = \frac{1}{\kappa W(1,\omega)} A(R_1(g),\omega) \quad (\omega \in \Omega_1)$$

Then

$$A^1 = \left((\Omega_1, \Sigma|\Omega_1, \mathbb{P}|\Omega_1), (A_\omega^1)_{\omega\in\Omega_1}\right),$$

with $\Sigma|\Omega_1$ the induced σ-algebra and $\mathbb{P}|\Omega_1$ the normalized restriction of $\mathbb{P}$, is a randomized algorithm for the approximation of the embedding $J : C^r(Q) \to C(Q)$ of cardinality $\le n$. From (94) we conclude

$$e_n(\mathscr{S}_1, R_1(B_{C^r(Q)}) \times \Omega, C(Q)) \ \ge\ c e_n^{\mathrm{ran}}(J, B_{C^r(Q)}, C(Q)) \ge c n^{-r/d}$$

(the latter relation being well-known, see [21], [12] for this result and also for the definition of the randomized n-th minimal errors e_n^{ran}). We conclude that

$$\begin{aligned}
&e_n(\mathscr{S}_1, F^{r,\varrho}(Q\times[0,1]\times\Omega;\kappa)\times\Omega, C(Q)) \\
&\quad\geq\ c e_n(\mathscr{S}_1, R_1(B_{C^r(Q)})\times\Omega, C(Q)) \geq c n^{-r/d},
\end{aligned}$$

and therefore, using (69),

$$\begin{aligned}
&e_n(\mathscr{S}, F_q^{r,\varrho}(Q\times[0,1]\times\Omega;\kappa)\times\Omega, C(Q\times[0,1])) \\
&\quad\geq\ e_n(\mathscr{S}, R_1(B_{C^r(Q)})\times\Omega, C(Q\times[0,1])) \\
&\quad\geq\ e_n(\mathscr{S}_1, R_1(B_{C^r(Q)})\times\Omega, C(Q)) \geq c n^{-r/d}.
\end{aligned}$$

Now we show that, similarly to the Banach space case, real-valued stochastic integration reduces to parametric stochastic integration. Let $y_0 \in C^r(Q)$, $y_0 \equiv 1$, let $V \in C(Q)^*$ be given by $\langle g, V\rangle = g(0)$, and $U \in \mathscr{L}(\mathbb{R}, C^r(Q))$ by $U\beta = \kappa\beta y_0$. Define

$$R_2 : B_{\mathscr{C}^\varrho([0,1])} \to F_q^{r,\varrho}(Q\times[0,1]\times\Omega;\kappa), \quad (R_2(f))(s,t,\omega) = \kappa f(t).$$

Let $f \in B_{\mathscr{C}^\varrho([0,1])}$. Then $\overline{R_2(f)} = Uf$. By (32) ,

$$V\mathscr{S}(R_2(f),\omega) \;=\; VS^J(Uf,\omega) = S^{VJU}(f,\omega) = \kappa S^{I_\mathbb{R}}(f,\omega) \quad (\mathbb{P}\text{-a.s.}),$$

and similarly,

$$V\mathscr{S}_1(R_2(f),\omega) \;=\; \kappa S_1^{I_\mathbb{R}}(f,\omega) \quad (\mathbb{P}\text{-a.s.}),$$

It is readily verified that an algorithm for $\mathscr{S}_1$ can be turned into an algorithm for $S_1^{I_\mathbb{R}}$ of the same cardinality, and similarly for $\mathscr{S}$ and $S^{I_\mathbb{R}}$. Following the same pattern of proof as in the Banach space case, we obtain, taking also into account (91),

$$\begin{aligned}
&e_n(\mathscr{S}_1, F^{r,\varrho}(Q\times[0,1]\times\Omega;\kappa)\times\Omega, C(Q)) \\
&\quad\geq\ \kappa e_n(S_1^{I_\mathbb{R}}, B_{\mathscr{C}^\varrho([0,1])}\times\Omega, \mathbb{R}) \geq c n^{-\varrho},
\end{aligned}$$

and respectively, using (92) and (93),

$$\begin{aligned}
&e_n(\mathscr{S}, F_q^{r,\varrho}(Q\times[0,1]\times\Omega;\kappa)\times\Omega, C(Q\times[0,1])) \\
&\quad\geq\ \kappa e_n(S^{I_\mathbb{R}}, B_{\mathscr{C}^\varrho([0,1])}\times\Omega, C([0,1])) \\
&\quad\geq\ c\max(n^{-\varrho}, n^{-1/2}(\log n)^{1/2}).
\end{aligned}$$

□

Acknowledgements. We want to thank Paweł Przybyłowicz and an anonymous referee for valuable comments which helped to improve the presentation.

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