Randomized Approximation of Sobolev Embeddings III

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Abstract

We continue the study of randomized approximation of embeddings between Sobolev spaces on the basis of function values. The source space is a Sobolev space with nonnegative smoothness order, the target space has negative smoothness order. The optimal order of approximation (in some cases only up to logarithmic factors) is determined. Extensions to Besov and Bessel potential spaces are given and a recently posed problem by Novak and Woźniakowski is partially solved. The results are applied to the complexity analysis of weak solution of elliptic PDE.

1 Introduction

In this paper we study randomized approximation of Sobolev embeddings $W_p^r(Q)$ into $W_q^s(Q)$, continuing the investigations from [10], where the case $s = 0$ and Q being a cube was considered, and from [11], concerned with the case $s \geq 0$, Q a bounded Lipschitz domain. Now we deal with the case $s < 0$, again in general Lipschitz domains Q. We determine the optimal order of randomized approximation based on function values (sometimes only up to logarithmic factors). The results are new even for the case of Q being a cube and $p = q = 2$.

The case $s < 0$ is of interest in view of its role for weak solution of elliptic partial differential equations. We present some consequences in this direction.

The paper is organized as follows. In section 3 we study the case $r = 0$. This is the essentially new situation, and we develop a multilevel Monte Carlo approximation algorithm. In section 4 we combine it with the algorithm from [11] to cover the case of general r. The deterministic setting is discussed in section 5, which also contains comparisons between the rates of deterministic and randomized approximation. In section 6 we extend the results to other types of function spaces, which leads, in particular, to the solution of open problem 25 of Novak and Woźniakowski [15] for the case of standard information. Finally, in section 7 an application to the complexity of weak solution of elliptic PDE is shown.

Many results are formulated in a slightly stronger way involving the dual of a Sobolev space with positive smoothness order as target space. These spaces are closely related to Sobolev spaces with negative smoothness order (see relation (127)), and the respective results for the latter are easily derived using duality (see Corollary 4.3 for Sobolev spaces and relations (171), (172), and Theorem 6.4 for the same situation in other function spaces).

2 Preliminaries

The paper is a direct continuation of [11]. Therefore we frequently use notation from there and refer to [11] for explanation. For $1 \le p \le \infty$ we denote by p^* the dual exponent given by $1/p + 1/p^* = 1$. For a normed space X we denote the unit ball by $\mathcal{B}_X$ and the dual space by X^*. Throughout this paper log means $\log_2$.

We need some results on Banach space valued random variables. Given p with $1 \le p \le 2$, we recall from Ledoux and Talagrand [12] that the type p constant $\tau_p(Z)$ of a Banach space Z is the smallest c with $0 < c \le +\infty$, such that for all n and all sequences $(z_i)_{i=1}^n \subset Z$,

$$\mathbb{E} \Big\| \sum_{i=1}^n \varepsilon_i z_i \Big\|^p \le c^p \sum_{i=1}^n \|z_i\|^p, \tag{1}$$

where (ε_i) denotes a sequence of independent symmetric Bernoulli random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$, i.e. $\mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = \frac{1}{2}$. Z is said to be of type p if $\tau_p(Z) < \infty$. Trivially, each Banach space is of type 1. Type p implies type p_1 for all $1 \le p_1 < p$. For $1 \le p < \infty$ all L_p spaces are of type $\min(p,2)$. Moreover, the spaces ℓ_p^n are of type $\min(p,2)$ uniformly in n, that is, $\tau_{\min(p,2)}(\ell_p^n) \le c$. Furthermore, $c_1(\log(n+1))^{1/2} \le \tau_2(\ell_\infty^n) \le c_2(\log(n+1))^{1/2}$.

We will use the following result. The case $p_1 = p$ of it is contained in Proposition 9.11 of [12]. The proof provided there easily extends to the case of general p_1 using some further tools from [12].

Lemma 2.1. *Let $1 \le p \le 2$, $p \le p_1 < \infty$. Then there is a constant $c > 0$ such that for each Banach space Z of type p, each $n \in \mathbb{N}$ and each sequence of independent, mean zero Z-valued random variables $(\zeta_i)_{i=1}^n$ with $\mathbb{E}\,\|\zeta_i\|^{p_1} < \infty$ $(1 \le i \le n)$ the following holds:*

$$\left(\mathbb{E} \Big\| \sum_{i=1}^n \zeta_i \Big\|^{p_1} \right)^{1/p_1} \le c\tau_p(Z) \left(\sum_{i=1}^n \left(\mathbb{E}\,\|\zeta_i\|^{p_1} \right)^{p/p_1} \right)^{1/p}.$$

Proof. Let $(\Omega, \Sigma, \mathbb{P})$ be the probability space the ζ_i are defined on. Let $(\varepsilon_i)_{i=1}^n$ be independent, symmetric Bernoulli random variables on some probability space $(\Omega', \Sigma', \mathbb{P}')$ different from $(\Omega, \Sigma, \mathbb{P})$. We denote the expectation with respect to $\mathbb{P}'$ by $\mathbb{E}'$ (and the expectation with respect to $\mathbb{P}$, as before, by $\mathbb{E}$). Using first Lemma 6.3 of [12] and then the equivalence of moments (Theorem 4.7 of [12]), we get

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{i=1}^n \zeta_i\right\|^{p_1}\right)^{1/p_1} &\leq 2\left(\mathbb{E}\,\mathbb{E}'\left\|\sum_{i=1}^n \varepsilon_i\zeta_i\right\|^{p_1}\right)^{1/p_1} \\ &\leq 2c_{p,p_1}\left(\mathbb{E}\left(\mathbb{E}'\left\|\sum_{i=1}^n \varepsilon_i\zeta_i\right\|^{p}\right)^{p_1/p}\right)^{1/p_1}, \end{aligned} \tag{2}$$

where the constant c_{p,p_1} depends only on p and p_1. Next we use the type inequality (1) and the triangle inequality in $L_{p/p_1}(\Omega, \mathbb{P})$ to obtain

$$\begin{aligned}\left(\mathbb{E}\left(\mathbb{E}'\left\|\sum_{i=1}^n \varepsilon_i\zeta_i\right\|^{p}\right)^{p_1/p}\right)^{1/p_1} &\leq \tau_p(Z)\left(\mathbb{E}\left(\sum_{i=1}^n \|\zeta_i\|^p\right)^{p_1/p}\right)^{1/p_1} \\ &\leq \tau_p(Z)\left(\sum_{i=1}^n \left(\mathbb{E}\,\|\zeta_i\|^{p_1}\right)^{p/p_1}\right)^{1/p}. \end{aligned} \tag{3}$$

Combining (2) and (3) completes the proof.

□

3 The case $r = 0$

Let $d \in \mathbb{N}$, let $Q \subset \mathbb{R}^d$ be a bounded Lipschitz domain (see [11], section 2 for details) and let $s \in \mathbb{N}_0$, $1 \leq p, q \leq \infty$. In this section, starting from an approximation of the embedding

$$J_{1,0} : W^s_{q^*}(Q) \to L_{p^*}(Q), \tag{4}$$

we produce and study an approximation of the mapping

$$J_1 = J_{1,0}^* I : L_p(Q) \to W^s_{q^*}(Q)^*, \tag{5}$$

where $J_{1,0}^*$ denotes the adjoint operator, and

$$I : L_p(Q) \to L_{p^*}(Q)^* \tag{6}$$

is the identity for $1 < p \leq \infty$ and the canonical embedding $L_1(Q) \to L_\infty(Q)^*$ for $p = 1$. In other words, $J_1 f$ is given for $f \in L_p(Q)$ by the relation

$$(J_1 f)(g) = \int_Q f(x)g(x)dx \quad (g \in W^s_{q^*}(Q)). \tag{7}$$

The Sobolev embedding theorem (see [1], Th. 5.4) states that, if

$$\left.\begin{array}{llll} & 1 < p \le \infty & & \text{and} \quad \frac{s}{d} \ge \left(\frac{1}{p} - \frac{1}{q}\right)_+ \\ \text{or} & & & \\ & p = 1, & 1 < q < \infty, & \text{and} \quad \frac{s}{d} > \frac{1}{q^*} \\ \text{or} & & & \\ & p = 1, & q \in \{1, \infty\}, & \text{and} \quad \frac{s}{d} \ge \frac{1}{q^*}, \end{array}\right\} \tag{8}$$

then the embedding of $W^s_{q^*}(Q)$ into $L_{p^*}(Q)$ is continuous, and hence, so is $J_1 : L_p(Q) \to W^s_{q^*}(Q)^*$. Here we used the notation $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$.

Let $\varrho \in \mathbb{N}_0$, $\varrho \ge s - 1$, let $\mathcal{P}_\varrho$ be the space of polynomials of degree not exceeding ϱ and let φ_j $(j = 1, \dots, \kappa)$ be any basis of $\mathcal{P}_\varrho$ which is orthonormal with respect to the $L_2([0,1]^d)$ scalar product. Let $P : L_1([0,1]^d) \to \mathcal{P}_\varrho$ be defined by

$$Pf = \sum_{j=1}^{\kappa} (f, \varphi_j)_{[0,1]^d}\, \varphi_j \quad (f \in L_1([0,1]^d)). \tag{9}$$

Here and below we use the notation

$$(g, h)_C := \int_C g(x)h(x)dx, \quad (g, h) := (g, h)_Q.$$

Clearly,

$$Pf = f \quad \text{for all} \quad f \in \mathcal{P}_\varrho. \tag{10}$$

Let $\tilde{Q}$ be any axis-parallel cube

$$\tilde{Q} = x_0 + [0, b]^d \quad \text{with} \quad Q \subset \tilde{Q}. \tag{11}$$

For $l \in \mathbb{N}_0$ let

$$\tilde{Q} = \bigcup_{i=1}^{2^{dl}} Q_{li},$$

where the Q_{li} are cubes of sidelength $b2^{-l}$ and of disjoint interior. Let x_{li} denote the point in Q_{li} with minimal coordinates. Let the scaling operators E_{li} and R_{li}, acting from $\mathcal{F}(\mathbb{R}^d)$, the space of all scalar functions on $\mathbb{R}^d$, to $\mathcal{F}(\mathbb{R}^d)$, be defined for $f \in \mathcal{F}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ by

$$(E_{li} f)(x) = f(x_{li} + b2^{-l}x) \tag{12}$$

and

$$(R_{li} f)(x) = f(b^{-1}2^l(x - x_{li})). \tag{13}$$

Clearly, (12) and (13) imply for $1 \le u \le \infty$ and $f \in L_u(\mathbb{R}^d)$, $g \in L_{u^*}(\mathbb{R}^d)$,

$$(E_{li} f, g)_{\mathbb{R}^d} = b^{-d} 2^{dl} (f, R_{li} g)_{\mathbb{R}^d}. \tag{14}$$

Define

$$\begin{aligned}\mathcal{I}_l &= \{i : 1 \le i \le 2^{dl},\ Q_{li} \subseteq Q\} \\ \mathcal{K}_l &= \{k : 1 \le k \le 2^{dl},\ Q_{lk} \cap Q \neq \emptyset\}.\end{aligned}$$

Thus, $\mathcal{I}_l$ is the set of indices of all 'small' cubes contained in Q, while $\mathcal{K}_l$ is the set of indices of all 'small' cubes intersecting Q. Put

$$Q_l = \bigcup_{k \in \mathcal{K}_l} Q_{lk}.$$

Note that

$$Q \subseteq Q_{l+1} \subseteq Q_l.$$

Lemma 3.1. *There are constants $a_0 > b\sqrt{d}$ and $l_0 \in \mathbb{N}_0$ such that for all $l \ge l_0$ and for all $k \in \mathcal{K}_l$ there is an $i \in \mathcal{I}_l$ such that*

$$Q_{lk} \subseteq B(x_{li}, a_0 2^{-l}).$$

This is a simple consequence of Lemma 3.2 in [11], where l_0 is the same as there and $a_0 = a + b\sqrt{d}$, with a the other constant from that lemma.

Using Lemma 3.1, we choose for $l \ge l_0$ any disjoint partition

$$\mathcal{K}_l = \bigcup_{i \in \mathcal{I}_l} \mathcal{K}_{li}$$

with

$$Q_{lk} \subseteq B(x_{li}, a_0 2^{-l}) \quad (k \in \mathcal{K}_{li}). \tag{15}$$

For $i \in \mathcal{I}_l$ put

$$\tilde{Q}_{li} = \bigcup_{k \in \mathcal{K}_{li}} Q_{lk}.$$

By definition,

$$Q_l = \bigcup_{i \in \mathcal{I}_l} \tilde{Q}_{li}, \tag{16}$$

and

$$\mu(\tilde{Q}_{li} \cap \tilde{Q}_{lj}) = 0 \quad (i \neq j \in \mathcal{I}_l), \tag{17}$$

with μ being the Lebesgue measure on $\mathbb{R}^d$. Summarizing, we have covered Q by μ-almost disjoint sets $\tilde{Q}_{li}$ each consisting of 'small' cubes close to x_{li}.

For $l \in \mathbb{N}_0$, $l \ge l_0$, define $\tilde{P}_l : L_1(Q) \to L_\infty(Q_l)$ by setting

$$\tilde{P}_l f = \sum_{i \in \mathcal{I}_l} \chi_{\tilde{Q}_{li}} R_{li} P E_{li} f \tag{18}$$

and $P_l : L_1(Q) \to L_\infty(Q)$ by restriction,

$$P_l f = (\tilde{P}_l f)|_Q. \tag{19}$$

We need P_l (more precisely, P_l^*) for the approximation of J_1, while $\tilde{P}_l$ will be used to derive certain estimates. Let $\mathcal{E} : W_{q^*}^s(Q) \to W_{q^*}^s(\mathbb{R}^d)$ be a bounded extension operator (see [16]).

Lemma 3.2. *Assume that the embedding condition (8) holds. Then for* $l \ge l_0$

$$\sup_{f\in\mathcal{B}_{W^s_{q^*}(Q)}} \|\mathcal{E}f - \tilde{P}_l f\|_{L_{p^*}(Q_l)} \quad\le\quad c\,2^{-sl+\max(1/p-1/q,0)dl}. \tag{20}$$

Proof. We denote $B = B^0(0, 2a_0/b)$, where a_0 is the constant from Lemma 3.1 and b from (11). By (8),

$$\|f\|_{L_{p^*}(B)} \le c\|f\|_{W^s_{q^*}(B)} \quad (f \in W^s_{q^*}(B)). \tag{21}$$

It follows from (9) and (21) that for $f \in W^s_{q^*}(B)$

$$\|Pf\|_{L_{p^*}(B)} \quad\le\quad c\|f\|_{W^s_{q^*}(B)}. \tag{22}$$

Let

$$|f|_{s,q^*,B} = \left(\sum_{|\alpha|=s} \|D^\alpha f\|^{q^*}_{L_{q^*}(B)}\right)^{1/q^*}$$

if $q^* < \infty$, and

$$|f|_{s,\infty,B} = \max_{|\alpha|=s} \|D^\alpha f\|_{L_\infty(B)}.$$

By Theorem 3.1.1 from [2] there is a constant $c > 0$ such that for all $f \in W^s_{q^*}(B)$

$$\inf_{g\in\mathcal{P}_\varrho} \|f-g\|_{W^s_{q^*}(B)} \le c|f|_{s,q^*,B}. \tag{23}$$

Consequently, using (10) and (21–23),

$$\begin{aligned} \|f - Pf\|_{L_{p^*}(B)} &= \inf_{g\in\mathcal{P}_\varrho} \|(f-g) - P(f-g)\|_{L_{p^*}(B)} \\ &\le c \inf_{g\in\mathcal{P}_\varrho} \|f-g\|_{W^s_{q^*}(B)} \le c|f|_{s,q^*,B}. \end{aligned} \tag{24}$$

Let $f \in W^s_{q^*}(Q)$. Denote $\tilde{f} = \mathcal{E}f$ and $B_{li} = B^0(x_{li}, a_0 2^{-l+1})$. We use the elementary relation

$$\|R_{li}g\|_{L_{p^*}(B_{li})} = b^{d/p^*} 2^{-dl/p^*} \|g\|_{L_{p^*}(B)} \quad (g \in L_{p^*}(B)). \tag{25}$$

From (16) and (17) we get

$$\begin{aligned} \|\mathcal{E}f - \tilde{P}_l f\|_{L_{p^*}(Q_l)} &= \Big\| \sum_{i\in\mathcal{I}_l} \chi_{\tilde{Q}_{li}} (\tilde{f} - R_{li}PE_{li}\tilde{f}) \Big\|_{L_{p^*}(Q_l)} \\ &= \left(\sum_{i\in\mathcal{I}_l} \|\tilde{f} - R_{li}PE_{li}\tilde{f}\|^{p^*}_{L_{p^*}(\tilde{Q}_{li})} \right)^{1/p^*}. \end{aligned} \tag{26}$$

Furthermore, using (15), (25), and (24),

$$\begin{aligned}
\|\tilde f - R_{li}PE_{li}\tilde f\|_{L_{p^*}(\tilde Q_{li})} &\le \|\tilde f - R_{li}PE_{li}\tilde f\|_{L_{p^*}(B_{li})} \\
&\le c\,2^{-dl/p^*}\|E_{li}\tilde f - PE_{li}\tilde f\|_{L_{p^*}(B)} \\
&\le c\,2^{-dl/p^*}|E_{li}\tilde f|_{s,q^*,B}. \qquad (27)
\end{aligned}$$

Arguing as in [11], relation (37), we obtain

$$\left(2^{-dl}\sum_{i\in\mathcal{I}_l}|E_{li}\tilde f|^{p^*}_{s,q^*,B}\right)^{1/p^*} \le c\,2^{-sl+\max(1/p-1/q,0)dl}\|f\|_{W^s_{q^*}(Q)}. \qquad (28)$$

Combining (26–28) gives

$$\|\mathcal{E}f - \tilde P_l f\|_{L_{p^*}(Q_l)} \le c\,2^{-sl+\max(1/p-1/q,0)dl}\|f\|_{W^s_{q^*}(Q)}.$$

□

Now we define $\tilde T_l : L_1(Q) \to L_\infty(Q_l)$ for $l \in \mathbb{N}_0$, $l \ge l_0$: If $l = l_0$, we put

$$\tilde T_{l_0} = \tilde P_{l_0}$$

and if $l \ge l_0 + 1$, we define $\tilde T_l$ by setting for $f \in L_1(Q)$

$$\tilde T_l f = \tilde P_l f - (\tilde P_{l-1} f)|_{Q_l}. \qquad (29)$$

Let $T_l : L_1(Q) \to L_\infty(Q)$ be given by

$$T_l f = (\tilde T_l f)|_Q = P_l f - P_{l-1} f. \qquad (30)$$

It follows that for any $L \in \mathbb{N}_0$, $L \ge l_0$,

$$P_L = \sum_{l=l_0}^{L} T_l. \qquad (31)$$

Put

$$n_l = \kappa|\mathcal{K}_l|, \qquad (32)$$

hence

$$c_1 2^{dl} \le n_l \le c_2 2^{dl}. \qquad (33)$$

Observe that the linear independence of $(\varphi_j)_{j=1}^{\kappa}$ and the disjointness of the interiors of the Q_{li} imply that for $1 \le u \le \infty$ there are constants $c_1, c_2 > 0$ such that for $b_{kj} \in \mathbb{K}$ ($k \in \mathcal{K}_l, j = 1, \dots, \kappa$)

$$c_1\|(b_{kj})\|_{\ell_u^{n_l}} \le 2^{dl/u}\left\|\sum_{k\in\mathcal{K}_l}\sum_{j=1}^{\kappa} b_{kj}\chi_{Q_{lk}}R_{lk}\varphi_j\right\|_{L_u(Q_l)} \le c_2\|(b_{kj})\|_{\ell_u^{n_l}}. \qquad (34)$$

Lemma 3.3. *We can represent* $\tilde{T}_l$ *in a unique way as*

$$\tilde{T}_l f = \sum_{k\in\mathcal{K}_l} \sum_{j=1}^{\kappa} (f, h_{lkj})\, \chi_{Q_{lk}} R_{lk}\varphi_j \quad (f \in L_1(Q)) \tag{35}$$

with

$$h_{lkj} \in L_\infty(Q). \tag{36}$$

Moreover, if (8) is fulfilled, then there are constants $c_1, c_2 > 0$ *such that for* $l \geq l_0$ *the following hold:*

$$\|\tilde{T}_l : W^s_{q^*}(Q) \to L_{p^*}(Q_l)\| \leq c_1 2^{-sl+\max(1/p-1/q,0)dl}, \tag{37}$$

and for all $a_{kj} \in \mathbb{K}$ $(k \in \mathcal{K}_l, j = 1, \ldots, \kappa)$

$$\Big\| \sum_{k\in\mathcal{K}_l} \sum_{j=1}^{\kappa} a_{kj} h_{lkj} \Big\|_{W^s_{q^*}(Q)^*} \leq c_2 2^{-sl+dl/p^*+\max(1/p-1/q,0)dl} \|(a_{kj})\|_{\ell_p^{n_l}}. \tag{38}$$

Proof. For $l \geq l_0$ and $k \in \mathcal{K}_l$ let $\iota(l,k)$ be the unique index $i \in \mathcal{I}_l$ with $k \in \mathcal{K}_{li}$. Let $f \in L_1(Q)$. Using (9), (18), and (14) we can represent $\tilde{P}_l f$ as

$$\begin{aligned} \tilde{P}_l f &= \sum_{i\in\mathcal{I}_l} \sum_{j=1}^{\kappa} (E_{li} f, \varphi_j)_{[0,1]^d}\, \chi_{\tilde{Q}_{li}} R_{li}\varphi_j \\ &= b^{-d} 2^{dl} \sum_{i\in\mathcal{I}_l} \sum_{k\in\mathcal{K}_{li}} \sum_{j=1}^{\kappa} (f, \chi_{Q_{li}} R_{li}\varphi_j)\, \chi_{Q_{lk}} R_{li}\varphi_j \\ &= b^{-d} 2^{dl} \sum_{k\in\mathcal{K}_l} \sum_{j=1}^{\kappa} (f, \chi_{Q_{l,\iota(l,k)}} R_{l,\iota(l,k)}\varphi_j)\, \chi_{Q_{lk}} R_{l,\iota(l,k)}\varphi_j. \end{aligned} \tag{39}$$

Since $(R_{lk}\varphi_j)_{j=1}^{\kappa}$ is a basis of $\mathcal{P}_\varrho(Q_{lk})$, the space of restrictions of polynomials from $\mathcal{P}_\varrho$ to Q_{lk}, we can express

$$\chi_{Q_{lk}} R_{l,\iota(l,k)}\varphi_j = \sum_{m=1}^{\kappa} \alpha_{lkjm} \chi_{Q_{lk}} R_{lk}\varphi_m$$

with $\alpha_{lkjm} \in \mathbb{K}$. Inserting this into (39), we get

$$\tilde{P}_l f = b^{-d} 2^{dl} \sum_{k\in\mathcal{K}_l} \sum_{m=1}^{\kappa} \Big(f, \sum_{j=1}^{\kappa} \alpha_{lkjm}\, \chi_{Q_{l,\iota(l,k)}} R_{l,\iota(l,k)}\varphi_j \Big)\, \chi_{Q_{lk}} R_{lk}\varphi_m. \tag{40}$$

This shows (35) and (36) for the case $l = l_0$, with

$$h_{l_0 km} = b^{-d} 2^{dl_0} \sum_{j=1}^{\kappa} \alpha_{l_0 kjm}\, \chi_{Q_{l_0,\iota(l_0,k)}} R_{l_0,\iota(l_0,k)}\varphi_j. \tag{41}$$

For $l \geq l_0 + 1$ and $k \in \mathcal{K}_l$ let $\mu(l,k)$ be the unique $m \in \mathcal{K}_{l-1}$ with $Q_{lk} \subset Q_{l-1,m}$. For brevity we write

$$\upsilon(l,k) = \iota(l-1, \mu(l,k)).$$

From (39),

$$\begin{aligned}
&\tilde{P}_{l-1} f|_{Q_l} \\
&= b^{-d} 2^{d(l-1)} \sum_{m \in \mathcal{K}_{l-1}} \sum_{j=1}^{\kappa} (f, \chi_{Q_{l-1,\iota(l-1,m)}} R_{l-1,\iota(l-1,m)} \varphi_j)\, \chi_{Q_{l-1,m} \cap Q_l} R_{l-1,\iota(l-1,m)} \varphi_j \\
&= b^{-d} 2^{d(l-1)} \sum_{k \in \mathcal{K}_l} \sum_{j=1}^{\kappa} (f, \chi_{Q_{l-1,\iota(l-1,\mu(l,k))}} R_{l-1,\iota(l-1,\mu(l,k))} \varphi_j)\, \chi_{Q_{lk}} R_{l-1,\iota(l-1,\mu(l,k))} \varphi_j \\
&= b^{-d} 2^{d(l-1)} \sum_{k \in \mathcal{K}_l} \sum_{j=1}^{\kappa} (f, \chi_{Q_{l-1,\upsilon(l,k)}} R_{l-1,\upsilon(l,k)} \varphi_j)\, \chi_{Q_{lk}} R_{l-1,\upsilon(l,k)} \varphi_j. \qquad (42)
\end{aligned}$$

Let $\beta_{lkjm} \in \mathbb{K}$ be such that

$$\chi_{Q_{lk}} R_{l-1,\upsilon(l,k)} \varphi_j = \sum_{m=1}^{\kappa} \beta_{lkjm} \chi_{Q_{lk}} R_{lk} \varphi_m.$$

Inserting into (42) gives

$$\tilde{P}_{l-1} f|_{Q_l} = b^{-d} 2^{d(l-1)} \sum_{k \in \mathcal{K}_l} \sum_{m=1}^{\kappa} \left(f, \sum_{j=1}^{\kappa} \beta_{lkjm}\, \chi_{Q_{l-1,\upsilon(l,k)}} R_{l-1,\upsilon(l,k)} \varphi_j \right) \chi_{Q_{lk}} R_{lk} \varphi_m,$$

which together with (40) implies (35) and (36) for $l \geq l_0 + 1$ with

$$\begin{aligned}
h_{lkm} &= b^{-d} 2^{dl} \sum_{j=1}^{\kappa} \alpha_{lkjm}\, \chi_{Q_{l,\iota(l,k)}} R_{l,\iota(l,k)} \varphi_j \\
&\quad - b^{-d} 2^{d(l-1)} \sum_{j=1}^{\kappa} \beta_{lkjm}\, \chi_{Q_{l-1,\upsilon(l,k)}} R_{l-1,\upsilon(l,k)} \varphi_j. \qquad (43)
\end{aligned}$$

Since, by (34), the system $\{\chi_{Q_{lk}} R_{lk} \varphi_j \,:\, k \in \mathcal{K}_l,\, 1 \leq j \leq \kappa\}$ is linearly independent, representation (35) is unique.

Now assume that (8) holds. From (29) and Lemma 3.2 we get for $f \in W^s_{q^*}(Q)$ and $l > l_0$

$$\begin{aligned}
\|\tilde{T}_l f\|_{L_{p^*}(Q_l)} &= \|(\tilde{P}_l - \tilde{P}_{l-1}) f\|_{L_{p^*}(Q_l)} \\
&\leq \|\mathcal{E} f - \tilde{P}_l f\|_{L_{p^*}(Q_l)} + \|\mathcal{E} f - \tilde{P}_{l-1} f\|_{L_{p^*}(Q_l)} \\
&\leq c\, 2^{-sl + \max(1/p - 1/q, 0) dl} \|f\|_{W^s_{q^*}(Q)}. \qquad (44)
\end{aligned}$$

This also holds for $l = l_0$, which follows from the boundedness of $\tilde{P}_{l_0} : W^s_{q^*}(Q) \to L_{p^*}(Q_{l_0})$. Thus (37) is proved.

To show (38), we estimate, using Hölder's inequality,

$$\begin{aligned}
&\Big\| \sum_{k\in\mathcal{K}_l} \sum_{j=1}^{\kappa} a_{kj} h_{lkj} \Big\|_{W^s_{q^*}(Q)^*} \\
&= \sup_{f\in\mathcal{B}_{W^s_{q^*}(Q)}} \Big| \sum_{k\in\mathcal{K}_l} \sum_{j=1}^{\kappa} a_{kj} (h_{lkj}, f) \Big| \\
&\le \|(a_{kj})\|_{\ell_p^{n_l}} \sup_{f\in\mathcal{B}_{W^s_{q^*}(Q)}} \|((f, h_{lkj}))_{k\in\mathcal{K}_l,\, 1\le j\le\kappa}\|_{\ell^{n_l}_{p^*}} . \qquad (45)
\end{aligned}$$

Furthermore, taking into acount (34), (35), and (37), we get for $f \in \mathcal{B}_{W^s_{q^*}(Q)}$

$$\begin{aligned}
\|((f, h_{lkj}))_{k\in\mathcal{K}_l,\, 1\le j\le\kappa}\|_{\ell^{n_l}_{p^*}} &\le c\, 2^{dl/p^*} \Big\| \sum_{k\in\mathcal{K}_l} \sum_{j=1}^{\kappa} (f, h_{lkj}) \chi_{Q_{lk}} R_{lk} \varphi_j \Big\|_{L_{p^*}(Q_l)} \\
&= c\, 2^{dl/p^*} \|\tilde{T}_l f\|_{L_{p^*}(Q_l)} \\
&\le c\, 2^{-sl+dl/p^*+\max(1/p-1/q,0)dl}. \qquad (46)
\end{aligned}$$

Combining (45) and (46) proves (38).

□

The functions h_{lkj} are crucial for the algorithm below. Relations (41) and (43) in the proof above supply more details of their structure.

It follows from (30) and (35) that

$$T_l f = \sum_{k\in\mathcal{K}_l} \sum_{j=1}^{\kappa} (f, h_{lkj})\, \chi_{Q_{lk}\cap Q} R_{lk} \varphi_j. \qquad (47)$$

Now we are ready to define the algorithm. Fix any numbers $L \in \mathbb{N}_0$, $L \ge l_0$, and $N_l \in \mathbb{N}$ $(l = l_0, \dots, L)$ (these are algorithm parameters, they will be specified in the proof of Proposition 3.6). For $g \in L_p(Q)$ we approximate $J_1 g \in W^s_{q^*}(Q)^*$ as follows:

$$J_1 g = J^*_{1,0} I g \approx P^*_L I g = \sum_{l=l_0}^{L} T^*_l I g, \qquad (48)$$

where we used (31). Let $\tilde{g}$ be the extension of g to $\tilde{Q}$ by zero. We have by (47) and (14),

$$\begin{aligned}
T^*_l I g &= \sum_{k\in\mathcal{K}_l} \sum_{j=1}^{\kappa} (g, \chi_{Q_{lk}\cap Q} R_{lk}\varphi_j) h_{lkj} = \sum_{k\in\mathcal{K}_l} \sum_{j=1}^{\kappa} (\tilde{g}, R_{lk}\varphi_j)_{Q_{lk}} h_{lkj} \\
&= b^d 2^{-dl} \sum_{k\in\mathcal{K}_l} \sum_{j=1}^{\kappa} (E_{lk}\tilde{g}, \varphi_j)_{[0,1]^d}\, h_{lkj}. \qquad (49)
\end{aligned}$$

Let $(\xi_{li})_{l=l_0,i=1}^{L,N_l}$ be independent uniformly distributed on $[0,1]^d$ random variables on some probability space $(\Omega,\Sigma,\mathbb{P})$ and put

$$\eta_{lkji} = b^d 2^{-dl}(E_{lk}\tilde{g})(\xi_{li})\varphi_j(\xi_{li}). \tag{50}$$

Then

$$\mathbb{E}\,\eta_{lkji} = b^d 2^{-dl}(E_{lk}\tilde{g},\varphi_j)_{[0,1]^d}, \tag{51}$$

and we approximate the scalar products in (49) by the standard Monte Carlo method

$$b^d 2^{-dl}(E_{lk}\tilde{g},\varphi_j)_{[0,1]^d} \approx N_l^{-1}\sum_{i=1}^{N_l}\eta_{lkji}. \tag{52}$$

Relations (48), (49), and (52) lead to the following algorithm. For $\omega\in\Omega$ we set

$$A_\omega^{(1)}(g) = \sum_{l=l_0}^{L} N_l^{-1}\sum_{i=1}^{N_l}\eta_{li} \tag{53}$$

with

$$\eta_{li} = \sum_{k\in\mathcal{K}_l}\sum_{j=1}^{\kappa}\eta_{lkji}h_{lkj}. \tag{54}$$

Written in more detail, we have

$$A_\omega^{(1)}(g) = b^d\sum_{l=l_0}^{L}2^{-dl}N_l^{-1}\sum_{k\in\mathcal{K}_l}\sum_{j=1}^{\kappa}\sum_{i=1}^{N_l}(\chi_Q\, g)(x_{lk}+b2^{-l}\xi_{li})\,\varphi_j(\xi_{li})\,h_{lkj}$$

(the h_{lkj} given by (41) and (43)). We set

$$A^{(1)} = \left(A_\omega^{(1)}\right)_{\omega\in\Omega}. \tag{55}$$

Clearly,

$$A^{(1)}\in\mathcal{A}_M^{\rm ran}(L_p(Q),W_{q^*}^s(Q)^*) \quad\text{with}\quad M=\kappa\sum_{l=l_0}^{L}|\mathcal{K}_l|N_l\le\kappa\sum_{l=l_0}^{L}2^{dl}N_l. \tag{56}$$

Define

$$\sigma(p) = \begin{cases} 1/2 & \text{if} \quad p=\infty \\ 0 & \text{if} \quad 1\le p<\infty, \end{cases} \tag{57}$$

$$\bar{p} = \min(p,2). \tag{58}$$

Lemma 3.4. *Assume that (8) holds and let $p_1<\infty$ be such that $1\le p_1\le p$. Then for $l_0\le l\le L$*

$$\sup_{g\in\mathcal{B}_{L_p(Q)}}\left(\mathbb{E}\left\|N_l^{-1}\sum_{i=1}^{N_l}(\mathbb{E}\,\eta_{li}-\eta_{li})\right\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1}$$

$$\le\ c(l+1)^{\sigma(p)}2^{-sl+\max(1/p-1/q,0)dl}N_l^{-(1-1/\bar{p})}. \tag{59}$$

Proof. We can assume without loss of generality that

$$p_1 \geq \bar{p}, \tag{60}$$

since the case $p_1 < \bar{p}$ then follows by Hölder's inequality. Let $g \in \mathcal{B}_{L_p(Q)}$. We set

$$\zeta_{lkji} = \mathbb{E}\,\eta_{lkji} - \eta_{lkji}, \quad \zeta_{li} = (\zeta_{lkji})_{k\in\mathcal{K}_l,\,1\leq j\leq\kappa} \in \ell_p^{n_l},$$

with n_l defined by (32). Then (38) of Lemma 3.3 gives

$$\begin{aligned} &\Big\|N_l^{-1}\sum_{i=1}^{N_l}(\mathbb{E}\,\eta_{li} - \eta_{li})\Big\|_{W^s_{q^*}(Q)^*} \\ &= N_l^{-1}\Big\|\sum_{k\in\mathcal{K}_l}\sum_{j=1}^{\kappa}\left(\sum_{i=1}^{N_l}\zeta_{lkji}\right)h_{lkj}\Big\|_{W^s_{q^*}(Q)^*} \\ &\leq c\,2^{-sl+dl/p^*+\max(1/p-1/q,0)dl}N_l^{-1}\Big\|\sum_{i=1}^{N_l}\zeta_{li}\Big\|_{\ell_p^{n_l}}. \end{aligned} \tag{61}$$

Consequently, taking into account that $\log(n_l+1) \leq c(l+1)$ and using (60), we get from Lemma 2.1

$$\left(\mathbb{E}\Big\|\sum_{i=1}^{N_l}\zeta_{li}\Big\|_{\ell_p^{n_l}}^{p_1}\right)^{1/p_1} \leq c(l+1)^{\sigma(p)}\left(\sum_{i=1}^{N_l}\left(\mathbb{E}\,\|\zeta_{li}\|_{\ell_p^{n_l}}^{p_1}\right)^{\bar{p}/p_1}\right)^{1/\bar{p}}, \tag{62}$$

where $\sigma(p)$ was defined by (57). Moreover, for $p < \infty$

$$\begin{aligned} \left(\mathbb{E}\,\|\zeta_{li}\|_{\ell_p^{n_l}}^{p_1}\right)^{1/p_1} &\leq \left(\mathbb{E}\,\|\zeta_{li}\|_{\ell_p^{n_l}}^{p}\right)^{1/p} \\ &= \Big\|\left((\mathbb{E}\,|\zeta_{lkji}|^p)^{1/p}\right)_{k\in\mathcal{K}_l,\,1\leq j\leq\kappa}\Big\|_{\ell_p^{n_l}}. \end{aligned} \tag{63}$$

Furthermore, we have for $p < \infty$

$$\begin{aligned} (\mathbb{E}\,|\zeta_{lkji}|^p)^{1/p} &= (\mathbb{E}\,|\mathbb{E}\,\eta_{lkji} - \eta_{lkji}|^p)^{1/p} \leq 2(\mathbb{E}\,|\eta_{lkji}|^p)^{1/p} \\ &= 2b^d 2^{-dl}\,(\mathbb{E}\,|\varphi_j(\xi_{li})(E_{lk}\tilde{g})(\xi_{li})|^p)^{1/p} \\ &\leq c\,2^{-dl}\|E_{lk}\tilde{g}\|_{L_p([0,1]^d)}. \end{aligned} \tag{64}$$

Combining (63) and (64), we obtain

$$\begin{aligned} \left(\mathbb{E}\,\|\zeta_{li}\|_{\ell_p^{n_l}}^{p_1}\right)^{1/p_1} &\leq c\,2^{-dl}\Big\|\left(\|E_{lk}\tilde{g}\|_{L_p([0,1]^d)}\right)_{k\in\mathcal{K}_l}\Big\|_{\ell_p^{|\mathcal{K}_l|}} \\ &= c\,2^{-dl+dl/p}\Big\|\left(\|\tilde{g}\|_{L_p(Q_{lk})}\right)_{k\in\mathcal{K}_l}\Big\|_{\ell_p^{|\mathcal{K}_l|}} \\ &\leq c\,2^{-dl/p^*}\|\tilde{g}\|_{L_p(\tilde{Q})} \leq c\,2^{-dl/p^*}. \end{aligned} \tag{65}$$

The estimates (63), (64), and (65) also hold for $p = \infty$, provided $(\mathbb{E}\,|\cdot|^p)^{1/p}$ is replaced by $\operatorname{ess\,sup}_{\omega\in\Omega}|\cdot|$. Relation (65) together with (62) gives

$$\left(\mathbb{E}\left\|\sum_{i=1}^{N_l}\zeta_{li}\right\|_{\ell_p^{n_l}}^{p_1}\right)^{1/p_1} \le c(l+1)^{\sigma(p)}2^{-dl/p^*}N_l^{1/\bar{p}}. \tag{66}$$

Joining (61) and (66) proves (59).

□

Let us introduce

$$\nu_0 = \nu_0(p,q) = \begin{cases} \min(p,q,2) & \text{if} \quad q<\infty \\ 1 & \text{if} \quad q=\infty. \end{cases} \tag{67}$$

Lemma 3.5. *Assume that (8) holds, let* p_1 *and* ν *be such that* $p_1 < \infty$, $1 \le p_1 \le p$, *and* $1 \le \nu \le \nu_0$. *Then*

$$\begin{aligned} &\sup_{g\in\mathcal{B}_{L_p(Q)}} \left(\mathbb{E}\,\|J_1 g - A_\omega^{(1)}(g)\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \\ &\le \; c\,2^{-sL+\max(1/p-1/q,0)dL} \\ &\quad +c\left(\sum_{l=l_0}^{L}(l+1)^{\nu\sigma(p)}2^{-\nu sl+\nu\max(1/p-1/q,0)dl}N_l^{-\nu(1-1/\bar{p})}\right)^{1/\nu}. \end{aligned} \tag{68}$$

Proof. It suffices to prove the case

$$p_1 \ge \nu, \tag{69}$$

the case $p_1 < \nu$ being, again, a consequence of Hölder's inequality. Let $g \in \mathcal{B}_{L_p(Q)}$. It follows from (49), (51), and (54) that

$$\mathbb{E}\,\eta_{li} = b^d 2^{-dl}\sum_{k\in\mathcal{K}_l}\sum_{j=1}^{\kappa}(E_{lk}\tilde{g},\varphi_j)_{[0,1]^d}\,h_{lkj} = T_l^* I g, \tag{70}$$

and hence, by (31) and (53),

$$\mathbb{E}\,A_\omega^{(1)}(g) = P_L^* I g.$$

We have

$$\begin{aligned} &\left(\mathbb{E}\,\|J_1 g - A_\omega^{(1)}(g)\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \\ &\le \; \|J_1 g - P_L^* I g\|_{W_{q^*}^s(Q)^*} + \left(\mathbb{E}\,\|P_L^* I g - A_\omega^{(1)}(g)\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1}. \end{aligned} \tag{71}$$

The first term can be estimated, using (20),

$$\begin{aligned}
\|J_1 g - P_L^* I g\|_{W_{q^*}^s(Q)^*} &= \sup_{f \in \mathcal{B}_{W_{q^*}^s(Q)}} |(f, J_{1,0}^* I g) - (f, P_L^* I g)| \\
&= \sup_{f \in \mathcal{B}_{W_{q^*}^s(Q)}} |(J_{1,0} f - P_L f, I g)| \\
&\le \|I g\|_{L_{p^*}(Q)^*} \sup_{f \in \mathcal{B}_{W_{q^*}^s(Q)}} \|f - P_L f\|_{L_{p^*}(Q)} \\
&\le c\, 2^{-sL + \max(1/p - 1/q, 0) dL}. \qquad (72)
\end{aligned}$$

Now we deal with the second term on the right-hand side of (71). Using (31) and (70), we obtain

$$\begin{aligned}
&\left(\mathbb{E}\, \|P_L^* I g - A_\omega^{(1)}(g)\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \\
&= \left(\mathbb{E}\left\|\sum_{l=l_0}^{L}\left(T_l^* I g - N_l^{-1}\sum_{i=1}^{N_l}\eta_{li}\right)\right\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \\
&= \left(\mathbb{E}\left\|\sum_{l=l_0}^{L}\zeta_l\right\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1}, \qquad (73)
\end{aligned}$$

where we defined for $l_0 \le l \le L$

$$\zeta_l = N_l^{-1}\sum_{i=1}^{N_l}(\mathbb{E}\,\eta_{li} - \eta_{li}).$$

The $(\zeta_l)_{l=l_0}^L$ are independent, mean zero, $W_{q^*}^s(Q)^*$-valued random variables. The space $W_{q^*}^s(Q)^*$ is of type ν_0, with ν_0 defined in (67). Indeed, if $\nu_0 = 1$, this is trivial. If $\nu_0 > 1$, we have $1 < q < \infty$. It follows from the definition that $W_{q^*}^s(Q)$ is isomorphic to a subspace of a space $L_{q^*}(\mu)$ for some measure μ. Consequently, $W_{q^*}^s(Q)^*$ is isomorphic to a quotient space of $L_q(\mu)$, and therefore of the same type $\min(q, 2)$ as $L_q(\mu)$ (see [12], p. 247). It follows that $W_{q^*}^s(Q)^*$ is also of type $\nu \le \nu_0$. By Lemma 2.1 and (69) we have

$$\left(\mathbb{E}\left\|\sum_{l=l_0}^{L}\zeta_l\right\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \le c\left(\sum_{l=l_0}^{L}\left(\mathbb{E}\,\|\zeta_l\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{\nu/p_1}\right)^{1/\nu}. \qquad (74)$$

According to Lemma 3.4

$$\begin{aligned}
\left(\mathbb{E}\,\|\zeta_l\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} &= \left(\mathbb{E}\left\|N_l^{-1}\sum_{i=1}^{N_l}(\mathbb{E}\,\eta_{li} - \eta_{li})\right\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \\
&\le c(l+1)^{\sigma(p)} 2^{-sl + \max(1/p - 1/q, 0) dl} N_l^{-(1-1/\bar{p})}. \qquad (75)
\end{aligned}$$

Combining (73), (74), and (75) leads to

$$\begin{aligned}
&\sup_{g\in\mathcal{B}_{L_p(Q)}} \left(\mathbb{E}\,\|P_L^* I g - A_\omega^{(1)}(g)\|^{p_1}_{W^s_{q^*}(Q)^*}\right)^{1/p_1} \\
&\le\ c\left(\sum_{l=l_0}^{L}(l+1)^{\nu\sigma(p)}2^{-\nu sl+\nu\max(1/p-1/q,0)dl}N_l^{-\nu(1-1/\bar{p})}\right)^{1/\nu},
\end{aligned}$$

which together with (71) and (72) implies (68). □

To state the next result put

$$\theta \;=\; \theta(s,p,q)=\frac{s}{d}-\left(\frac{1}{p}-\frac{1}{q}\right)_+,\quad \tau=\tau(p)=1-\frac{1}{\bar{p}}, \tag{76}$$

$$\nu_1 \;=\; \nu_1(s,p,q)=$$

$$\begin{cases} 0 & \text{if}\quad \theta>\tau \\ 1 & \text{if}\quad \theta=\tau \quad\text{and}\quad p\le q<\infty \\ 2-1/\bar{p} & \text{if}\quad \theta=\tau \quad\text{and}\quad p<q=\infty \\ 2 & \text{if}\quad \theta=\tau \quad\text{and}\quad p=q=\infty \\ 1 & \text{if}\quad \theta=\tau \quad\text{and}\quad p>q \\ 0 & \text{if}\quad \theta<\tau \quad\text{and}\quad \min(p,q)<\infty \\ \theta & \text{if}\quad \theta<\tau \quad\text{and}\quad p=q=\infty, \end{cases} \tag{77}$$

where we recall that $\bar{p}=\min(p,2)$.

Proposition 3.6. *Let Q be a bounded Lipschitz domain, $s\in\mathbb{N}_0$, $1\le p,q\le\infty$, and assume that (8) holds. Let $1\le p_1<\infty$, $p_1\le p$. Then there are constants $c_1\in\mathbb{N}$, $c_2>0$ such that for each $n\in\mathbb{N}$ with $n\ge 2$ there is a choice of parameters $L,(N_l)_{l=l_0}^L$ such that algorithm $A^{(1)}$ belongs to $\mathcal{A}^{\mathrm{ran}}_{c_1 n}(L_p(Q),W^s_{q^*}(Q)^*)$ and the error satisfies*

$$\sup_{g\in\mathcal{B}_{L_p(Q)}} \left(\mathbb{E}\,\|J_1 g - A_\omega^{(1)}(g)\|^{p_1}_{W^s_{q^*}(Q)^*}\right)^{1/p_1} \le c_2 n^{-\min(\theta,\tau)}(\log n)^{\nu_1}.$$

Proof. Let $n\in\mathbb{N}$, $n\ge 2$. We put

$$N_l=\left\lceil L^{-\delta_0}2^{-dl-\delta_1 l-\delta_2(L-l)}n\right\rceil \quad (l=l_0,\dots,L),$$

with $L\in\mathbb{N}$, $L\ge l_0$, $\delta_0,\delta_1,\delta_2\ge 0$ to be fixed later. Then Lemma 3.5 gives for any $1\le\nu\le\nu_0$

$$\begin{aligned}
&\sup_{g\in\mathcal{B}_{L_p(Q)}} \left(\mathbb{E}\,\|J_1 g - A_\omega^{(1)}(g)\|^{p_1}_{W^s_{q^*}(Q)^*}\right)^{1/p_1} \\
&\le\ c\,2^{-sL+\max(1/p-1/q,0)dL}+cn^{-(1-1/\bar{p})}L^{(1-1/\bar{p})\delta_0}\left(\sum_{l=l_0}^{L}(l+1)^{\nu\sigma(p)}2^{\nu\lambda(l)}\right)
\end{aligned} \tag{78}$$

with

$$\lambda(l) = -sl + \left(\frac{1}{p} - \frac{1}{q}\right)_+ dl + \left(1 - \frac{1}{\bar{p}}\right)(dl + \delta_1 l + \delta_2(L - l)). \tag{79}$$

We distinguish between three cases. First we assume

$$\frac{s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+ > 1 - \frac{1}{\bar{p}} .$$

We put

$$L = \max\left(\left\lceil \frac{\log n}{d} \right\rceil, l_0\right), \tag{80}$$

$\delta_0 = \delta_2 = 0$, and choose any $\delta_1 > 0$ with

$$\frac{s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+ > \left(1 - \frac{1}{\bar{p}}\right)\left(1 + \frac{\delta_1}{d}\right).$$

Then

$$\sum_{l=l_0}^{L} (l+1)^{\sigma(p)} 2^{\lambda(l)} \le c, \tag{81}$$

moreover,

$$2^{-sL+\max(1/p-1/q,0)dL} \le cn^{-s/d+\max(1/p-1/q,0)} \le cn^{-(1-1/\bar{p})}, \tag{82}$$

and we get from (78) with $\nu = 1$, (81), and (82)

$$\sup_{g \in \mathcal{B}_{L_p(Q)}} \left(\mathbb{E}\, \| J_1 g - A_\omega^{(1)}(g) \|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \le cn^{-(1-1/\bar{p})}.$$

Furthermore, the number of sampling points, see (56), is

$$M \le \kappa \sum_{l=l_0}^{L} 2^{dl} N_l \le \kappa \sum_{l=l_0}^{L} 2^{dl}\left(2^{-dl-\delta_1 l} n + 1\right) \le c(n + 2^{dL}) \le cn.$$

This proves Proposition 3.6 in the case $\theta > \tau$.

Next let

$$\frac{s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+ = 1 - \frac{1}{\bar{p}} .$$

Here we take the same choice (80) of L, put $\delta_0 = 1$, $\delta_1 = \delta_2 = 0$, and conclude from (79) that $\lambda(l) = 0$, hence, by (78) with $\nu = \nu_0$,

$$\begin{aligned} &\sup_{g \in \mathcal{B}_{L_p(Q)}} \left(\mathbb{E}\, \| J_1 g - A_\omega^{(1)}(g) \|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \\ &\le \; cn^{-(1-1/\bar{p})} (\log n)^{1-1/\bar{p}+1/\nu_0+\sigma(p)}. \end{aligned} \tag{83}$$

The number of sampling points is

$$M \leq \kappa \sum_{l=l_0}^{L} 2^{dl} N_l \leq \kappa \sum_{l=l_0}^{L} 2^{dl} \left(L^{-1} 2^{-dl} n + 1 \right) \leq c(n + 2^{dL}) \leq cn.$$

If $p \leq q < \infty$, then by (57) and (67), $\nu_0 = \bar{p}$ and $\sigma(p) = 0$. If $p < q = \infty$, we have $\nu_0 = 1$ and $\sigma(p) = 0$, while in the case $p = q = \infty$ we get $\bar{p} = 2$, $\nu_0 = 1$, and $\sigma(p) = 1/2$. Inserting this into (83) proves the statement for the case $\theta = \tau$, $p \leq q$. The case $\theta = \tau$, $p > q$ and is considered later on.

Now assume

$$\frac{s}{d} - \left(\frac{1}{p} - \frac{1}{q} \right)_+ < 1 - \frac{1}{\bar{p}} ,$$

which together with (8) implies $\bar{p} > 1$, hence $\tau > 0$. We put

$$L = \max \left(\left\lceil \frac{\log n - \frac{\sigma(p)}{\tau} \log \log n}{d} \right\rceil , l_0 \right) .$$

This means

$$c_1 n (\log n)^{-\sigma(p)/\tau} \leq 2^{dL} \leq c_2 n (\log n)^{-\sigma(p)/\tau}.$$

Let $\delta_0 = \delta_1 = 0$ and let $\delta_2 > 0$ satisfy

$$\frac{s}{d} - \left(\frac{1}{p} - \frac{1}{q} \right)_+ < \left(1 - \frac{1}{\bar{p}} \right) \left(1 - \frac{\delta_2}{d} \right) .$$

Consequently, we have

$$\sum_{l=l_0}^{L} (l+1)^{\sigma(p)} 2^{\lambda(l)} \leq c L^{\sigma(p)} 2^{\lambda(L)}.$$

Moreover,

$$\lambda(L) = -sL + \left(\frac{1}{p} - \frac{1}{q} \right)_+ dL + \left(1 - \frac{1}{\bar{p}} \right) dL = -(\theta - \tau) dL$$

and hence, by (78), with $\nu = 1$,

$$\begin{aligned}
& \sup_{g \in \mathcal{B}_{L_p(Q)}} \left(\mathbb{E} \, \| J_1 g - A_\omega^{(1)}(g) \|_{W_{q^*}^s(Q)^*}^{p_1} \right)^{1/p_1} \\
\leq \quad & c \, 2^{-sL + \max(1/p - 1/q, 0) dL} + c n^{-(1 - 1/\bar{p})} L^{\sigma(p)} 2^{\lambda(L)} \\
= \quad & c \, 2^{-\theta dL} + c n^{-\tau} L^{\sigma(p)} 2^{-(\theta - \tau) dL} \\
\leq \quad & c n^{-\theta} (\log n)^{\theta \sigma(p)/\tau} + c n^{-\tau} (\log n)^{\sigma(p)} n^{-(\theta - \tau)} (\log n)^{(\theta - \tau)\sigma(p)/\tau} \\
\leq \quad & c n^{-\theta} (\log n)^{\theta \sigma(p)/\tau}. \qquad (84)
\end{aligned}$$

The number of sampling points can be estimated as

$$M \le \kappa \sum_{l=l_0}^{L} 2^{dl} N_l \le c \sum_{l=l_0}^{L} 2^{dl} \left(n 2^{-dl-\delta_2(L-l)} + 1 \right) \le c(n + 2^{dL}) \le cn.$$

This proves the case $\theta < \tau$, except for the subcase $q < p = \infty$.

Finally, we consider the two remaining situations

$$\theta = \tau, \quad p > q \tag{85}$$

and

$$\theta < \tau, \quad p = \infty > q. \tag{86}$$

By Hölder's inequality, we can assume the following: If $p = \infty$, then $p_1 > \max(q, 2)$, and if $p < \infty$, then $p_1 = p$. Consequently,

$$p \ge p_1 > q. \tag{87}$$

We factorize

$$J_1 : L_p(Q) \xrightarrow{J_{1,1}} L_{p_1}(Q) \xrightarrow{J_{1,2}} W^s_{p_1^*}(Q)^* \xrightarrow{J_{1,3}} W^s_{q^*}(Q)^* \tag{88}$$

with $J_{1,1}, J_{1,2}, J_{1,3}$ the respective embeddings, and use (83) and (84) with p_1 instead of p and q. Because of (76) and (87) we have $\theta(s, p, q) = \theta(s, p_1, p_1)$. Furthermore, in each of the choices of p_1 above we have $\bar{p} = \bar{p}_1$, hence, by (76), $\tau(p) = \tau(p_1)$. Finally, $p_1 < \infty$, so $\sigma(p_1) = 0$, and by (67), $\nu_0(p_1, p_1) = \bar{p}_1$.

Let $\tilde{A}^{(1)}$ denote algorithm $A^{(1)}$, considered as an element of

$$\mathcal{A}^{\text{ran}}(L_{p_1}(Q), W^s_{p_1^*}(Q)^*).$$

Then

$$A^{(1)}_\omega(g) = J_{1,3} \tilde{A}^{(1)}_\omega(J_{1,1} g) \quad (g \in L_p(Q)).$$

In the case (85) we get from (83)

$$\begin{aligned}
&\sup_{g \in \mathcal{B}_{L_p(Q)}} \left(\mathbb{E} \, \| J_1 g - A^{(1)}_\omega(g) \|^{p_1}_{W^s_{q^*}(Q)^*} \right)^{1/p_1} \\
&= \sup_{g \in \mathcal{B}_{L_p(Q)}} \left(\mathbb{E} \, \| J_{1,3} J_{1,2} J_{1,1} g - J_{1,3} \tilde{A}^{(1)}_\omega(J_{1,1} g) \|^{p_1}_{W^s_{q^*}(Q)^*} \right)^{1/p_1} \\
&\le \|J_{1,3}\| \sup_{g \in \mathcal{B}_{L_{p_1}(Q)}} \left(\mathbb{E} \, \| J_{1,2} g - \tilde{A}^{(1)}_\omega(g) \|^{p_1}_{W^s_{p_1^*}(Q)^*} \right)^{1/p_1} \|J_{1,1}\| \\
&\le c n^{-(1-1/\bar{p}_1)} (\log n)^{1-1/\bar{p}_1 + 1/\nu_0(p_1,p_1) + \sigma(p_1)} = c n^{-\tau(p)} \log n,
\end{aligned}$$

and in the case (86) from (84)

$$\begin{aligned}
&\sup_{g\in\mathcal{B}_{L_p(Q)}}\left(\mathbb{E}\,\|J_1 g - A_\omega^{(1)}(g)\|^{p_1}_{W^s_{q^*}(Q)^*}\right)^{1/p_1} \\
&\le \|J_{1,3}\| \sup_{g\in\mathcal{B}_{L_{p_1}(Q)}}\left(\mathbb{E}\,\|J_{1,2} g - \tilde{A}_\omega^{(1)}(g)\|^{p_1}_{W^s_{p_1^*}(Q)^*}\right)^{1/p_1}\|J_{1,1}\| \\
&\le cn^{-\theta(s,p_1,p_1)}(\log n)^{\theta(s,p_1,p_1)\sigma(p_1)/\tau(p_1)} = cn^{-\theta(s,p,q)}.
\end{aligned}$$

□

4 Main results

Let

$$r, s \in \mathbb{N}_0, \quad 1 \le p, q \le \infty. \tag{89}$$

Now we study approximation of the embedding

$$J : W^r_p(Q) \to W^s_{q^*}(Q)^*. \tag{90}$$

defined for $f \in W^r_p(Q)$ by the relation

$$(Jf)(g) = \int_Q f(x)g(x)dx \quad (g \in W^s_{q^*}(Q)). \tag{91}$$

First we state conditions under which it is well-defined and continuous. The Sobolev embedding theorem (see [1], Th. 5.4) affirms that $W^r_p(Q)$ is continuously embedded into $L_q(Q)$ if

$$\left.\begin{array}{llll}
 & 1 \le q < \infty & & \text{and} \quad \frac{r}{d} \ge \left(\frac{1}{p} - \frac{1}{q}\right)_+ \\
\text{or} & & & \\
 & q = \infty, & 1 < p < \infty, & \text{and} \quad \frac{r}{d} > \frac{1}{p} \\
\text{or} & & & \\
 & q = \infty, & p \in \{1, \infty\}, & \text{and} \quad \frac{r}{d} \ge \frac{1}{p}.
\end{array}\right\} \tag{92}$$

Recall also statement (8), which gives sufficient conditions for the continuity of the embedding of $W^s_{q^*}(Q)$ into $L_{p^*}(Q)$, and hence, by passing to the dual mapping, also of the embedding of $L_p(Q)$ into $W^s_{q^*}(Q)^*$. Let us formulate the following two conditions

$$r = 0,\ p = 1,\ 1 < q < \infty, \tag{93}$$

$$s = 0,\ q = \infty,\ 1 < p < \infty. \tag{94}$$

Then the embedding $J : W_p^r(Q) \to W_{q^*}^s(Q)^*$ is well-defined and continuous if

$$\left.\begin{array}{ll} & (93) \text{ holds and } \frac{s}{d} > \frac{1}{q^*}, \\ \text{or} & \\ & (94) \text{ holds and } \frac{r}{d} > \frac{1}{p}, \\ \text{or} & \\ & (93) \text{ and } (94) \text{ do not hold, and } \frac{r+s}{d} \geq \left(\frac{1}{p} - \frac{1}{q}\right)_+ . \end{array}\right\} \tag{95}$$

This is easily derived directly from (92) and (8). We do not give details since the continuity of J is also a by-product of the factorization of J in the proof of Proposition 4.1.

To approximate J, let $n \in \mathbb{N}$, $n \geq 2$, let

$$A^{(0)} = \left(P_{k,\omega_0}^{(0)}\right)_{\omega_0 \in \Omega_0}$$

be the algorithm defined in (23–25) of [11], with parameter k and $(\Omega_0, \Sigma_0, \mathbb{P}_0)$ the associated probability space. Let

$$A^{(1)} = \left(A_{\omega_1}^{(1)}\right)_{\omega_1 \in \Omega_1}$$

be the algorithm defined in (53–55), with parameters L, $(N_l)_{l_0}^L$, and probability space $(\Omega_1, \Sigma_1, \mathbb{P}_1)$. We combine both algorithms in the following way. Let

$$(\Omega, \Sigma, \mathbb{P}) = (\Omega_0, \Sigma_0, \mathbb{P}_0) \times (\Omega_1, \Sigma_1, \mathbb{P}_1)$$

and define an algorithm $A = (A_\omega)_{\omega \in \Omega}$ by setting for $\omega = (\omega_0, \omega_1)$ and $f \in \mathcal{F}(Q)$

$$A_\omega(f) = P_{k,\omega_0}^{(0)} f + A_{\omega_1}^{(1)}(f - P_{k,\omega_0}^{(0)} f) \tag{96}$$

(note that $P_{k,\omega_0}^{(0)} f \in \mathcal{F}(Q)$). Measurability and consistency follow from the definitions of $A^{(0)}$ and $A^{(1)}$, and we have

$$A \in \mathcal{A}^{\text{ran}}(W_p^r(Q), W_{q^*}^s(Q)^*). \tag{97}$$

Proposition 4.1. *Let $r, s \in \mathbb{N}_0$, $1 \leq p, q \leq \infty$, assume that (95) holds, and let $1 \leq p_1 < \infty$, $p_1 \leq p$. Then J is continuous and there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ with $n \geq 2$ there is a choice of parameters k, L, $(N_l)_{l=l_0}^L$ such that algorithm A belongs to $\mathcal{A}_{c_1 n}^{\text{ran}}(W_p^r(Q), W_{q^*}^s(Q)^*)$ and the error satisfies*

$$\sup_{f \in \mathcal{B}_{W_p^r(Q)}} \left(\mathbb{E}\, \|Jf - A_\omega(f)\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \leq c_2 n^{-\gamma} (\log n)^{\nu_1},$$

where

$$\gamma = \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d} + 1 - \frac{1}{\bar{p}}\right), \tag{98}$$

and ν_1 is given by (77).

Proof. In each of the cases considered in this proof we will find a number $1 \le w \le \infty$ such that (92) holds for the index pair (p, w) (meaning that (92) holds with q replaced by w) and (8) holds for the pair (w, q). Hence, both embeddings J_0 and J_1 in the factorization of J as

$$J : W_p^r(Q) \xrightarrow{J_0} L_w(Q) \xrightarrow{J_1} W_{q^*}^s(Q)^* \tag{99}$$

are continuous, and so is J.

Let $n \in \mathbb{N}$, $n \ge 2$. Now we fix the parameters in the definition (96) of algorithm A. We put

$$k = \max\left(\left\lceil \frac{\log n}{d} \right\rceil, l_0\right), \tag{100}$$

(recall the remark made after Lemma 3.1 that l_0 in the present paper is the same as in [11]), and let the parameters L, $(N_l)_{l=l_0}^L$ for $A^{(1)}$ be chosen according to Proposition 3.6, with the given n and the index pair (w, q). Hence

$$A \in \mathcal{A}_{cn}^{\text{ran}}(W_p^r(Q), W_{q^*}^s(Q)^*). \tag{101}$$

Let $1 \le t < \infty$. For fixed $f \in \mathcal{B}_{W_p^r(Q)}$ and $\omega_0 \in \Omega_0$ the linearity of $A_{\omega_1}^{(1)}$ gives

$$\begin{aligned}
&\mathbb{E}_{\omega_1} \|Jf - A_{(\omega_0,\omega_1)}(f)\|_{W_{q^*}^s(Q)^*}^t \\
&= \mathbb{E}_{\omega_1} \left\| J_1\left(J_0 f - P_{k,\omega_0}^{(0)} f\right) - A_{\omega_1}^{(1)}\left(J_0 f - P_{k,\omega_0}^{(0)} f\right) \right\|_{W_{q^*}^s(Q)^*}^t \\
&\le \left\| J_0 f - P_{k,\omega_0}^{(0)} f \right\|_{L_w(Q)}^t \sup_{g \in \mathcal{B}_{L_w(Q)}} \mathbb{E}_{\omega_1} \left\| J_1 g - A_{\omega_1}^{(1)}(g) \right\|_{W_{q^*}^s(Q)^*}^t .
\end{aligned}$$

This together with Fubini's theorem yields

$$\begin{aligned}
&\sup_{f \in \mathcal{B}_{W_p^r(Q)}} \left(\mathbb{E} \|Jf - A_\omega(f)\|_{W_{q^*}^s(Q)^*}^t\right)^{1/t} \\
&= \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \left(\mathbb{E}_{\omega_0} \mathbb{E}_{\omega_1} \|Jf - A_{(\omega_0,\omega_1)}(f)\|_{W_{q^*}^s(Q)^*}^t\right)^{1/t} \\
&\le \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \left(\mathbb{E}_{\omega_0} \left\| J_0 f - P_{k,\omega_0}^{(0)} f \right\|_{L_w(Q)}^t\right)^{1/t} \\
&\qquad \times \sup_{g \in \mathcal{B}_{L_w(Q)}} \left(\mathbb{E}_{\omega_1} \|J_1 g - A_{\omega_1}^{(1)}(g)\|_{W_{q^*}^s(Q)^*}^t\right)^{1/t} .
\end{aligned} \tag{102}$$

Case 1. Assume that (8) holds. We choose $w = p$ in the factorization (99), get from Proposition 3.3 of [11], using $p_1 \le p$,

$$\sup_{f \in \mathcal{B}_{W_p^r(Q)}} \left(\mathbb{E}_{\omega_0} \left\| J_0 f - P_{k,\omega_0}^{(0)} f \right\|_{L_p(Q)}^{p_1}\right)^{1/p_1} \le c n^{-r/d} \tag{103}$$

and from Proposition 3.6 of the present paper

$$\sup_{g\in\mathcal{B}_{L_p(Q)}} \left(\mathbb{E}_{\omega_1}\|J_1 g - A_{\omega_1}^{(1)}(g)\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \leq cn^{-\min\left(\frac{s}{d}-\left(\frac{1}{p}-\frac{1}{q}\right)_+,1-\frac{1}{\bar{p}}\right)}(\log n)^{\nu_1}. \tag{104}$$

Combining (98), (102), (103), and (104), we derive

$$\sup_{f\in\mathcal{B}_{W_p^r(Q)}} \left(\mathbb{E}\,\|Jf - A_\omega(f)\|_{W_{q^*}^s(Q)^*}^{p_1}\right)^{1/p_1} \leq cn^{-\gamma}(\log n)^{\nu_1},$$

which is the needed estimate in case of (8).

Now assume that (8) does not hold. This means that either

$$\frac{s}{d} = \frac{1}{q^*}, \quad p = 1, \quad 1 < q < \infty, \tag{105}$$

or

$$\frac{s}{d} < \frac{1}{p} - \frac{1}{q}. \tag{106}$$

Case 2. We assume (105). Together with (95) this implies $r > 0$ and hence we can find a $w > 0$ with

$$\frac{1}{p} - \frac{r}{d} = 1 - \frac{r}{d} < \frac{1}{w} < 1 = \frac{s}{d} + \frac{1}{q}, \quad p = 1 < w < \min(q, 2). \tag{107}$$

It follows that (92) is satisfied for the pair (p, w) and (8) for the pair (w, q). We have by Proposition 3.3 of [11],

$$\sup_{f\in\mathcal{B}_{W_p^r(Q)}} \left(\mathbb{E}_{\omega_0}\left\|J_0 f - P_{k,\omega_0}^{(0)} f\right\|_{L_w(Q)}^w\right)^{1/w} \leq cn^{-r/d+1/p-1/w}. \tag{108}$$

Next we consider the parameters involved into Proposition 3.6 above, for the pair (w, q). Inserting into (58) and (76), we get $\bar{w} = w$ and

$$\theta(s, w, q) = \frac{s}{d} - \left(\frac{1}{w} - \frac{1}{q}\right)_+ = \frac{s}{d} - \frac{1}{w} + \frac{1}{q} = 1 - \frac{1}{w} = 1 - \frac{1}{\bar{w}} = \tau(w).$$

Since $w < q < \infty$, we have by (77),

$$\nu_1(s, w, q) = 1.$$

Thus, Proposition 3.6 yields

$$\sup_{g\in\mathcal{B}_{L_w(Q)}} \left(\mathbb{E}_{\omega_1}\|J_1 g - A_{\omega_1}^{(1)}(g)\|_{W_{q^*}^s(Q)^*}^w\right)^{1/w} \leq cn^{-s/d+1/w-1/q}\log n. \tag{109}$$

Combining (102), (108), and (109) gives

$$\sup_{f\in\mathcal{B}_{W_p^r(Q)}} \left(\mathbb{E}\,\|Jf - A_\omega(f)\|^w_{W^s_{q^*}(Q)^*}\right)^{1/w} \;\le\; cn^{-(r+s)/d+1/p-1/q}\log n,$$

which proves the result for the case (105) since $p_1 = p = 1 < w$ and, by (76), (77), and (105), $\nu_1(s,p,q) = 1$.

Case 3. Now suppose (106) holds. We choose w so that

$$\frac{1}{w} = \frac{s}{d} + \frac{1}{q} < \frac{1}{p}, \tag{110}$$

thus $1 \le p < w \le q$ and (8) is satisfied for the pair (w,q). Moreover, (76) gives

$$\theta(s,w,q) = \frac{s}{d} - \left(\frac{1}{w} - \frac{1}{q}\right)_+ = 0, \quad \tau(w) = 1 - \frac{1}{\bar{w}} > 0.$$

By (77), $\nu_1(s,w,q) = 0$, so we obtain from Proposition 3.6

$$\sup_{g\in\mathcal{B}_{L_w(Q)}} \left(\mathbb{E}_{\,\omega_1}\|J_1 g - A^{(1)}_{\omega_1}(g)\|^{p_1}_{W^s_{q^*}(Q)^*}\right)^{1/p_1} \le c, \tag{111}$$

since $p_1 \le p < w$. Furthermore, by assumption (95),

$$\frac{r}{d} \ge \frac{1}{p} - \frac{1}{q} - \frac{s}{d} = \frac{1}{p} - \frac{1}{w}. \tag{112}$$

Now we show that (92) is fulfilled for (p,w). Indeed, if $w = \infty$ and $p = 1$ or if $w < \infty$, then this follows from (112). If $w = \infty$ and $1 < p < \infty$, then we note that (110) implies $s = 0$ and $q = \infty$, therefore (94) holds, so (95) gives

$$\frac{r}{d} > \frac{1}{p},$$

and thus, (92) for (p,w), again.

Consequently, by Proposition 3.3 of [11], using $p_1 < w$,

$$\sup_{f\in\mathcal{B}_{W_p^r(Q)}} \left(\mathbb{E}_{\,\omega_0}\left\|J_0 f - P^{(0)}_{k,\omega_0} f\right\|^{p_1}_{L_w(Q)}\right)^{1/p_1} \;\le\; cn^{-r/d+1/p-1/w}. \tag{113}$$

Taking into account (102), (111), and (113), we conclude

$$\begin{aligned}\sup_{f\in\mathcal{B}_{W_p^r(Q)}} \left(\mathbb{E}\,\|Jf - A_\omega(f)\|^{p_1}_{W^s_{q^*}(Q)^*}\right)^{1/p_1} &\le cn^{-r/d+1/p-1/w} \\ &= cn^{-(r+s)/d+1/p-1/q},\end{aligned}$$

which shows the result for the case (106).

□

Let $\widetilde{W}^s_{q^*}(Q)$ be the closure in the norm of $W^s_{q^*}(Q)$ of the set of C^∞ functions whose support is contained in Q and let $U : \widetilde{W}^s_{q^*}(Q) \to W^s_{q^*}(Q)$ be the identical embedding. Clearly,

$$\|U\| = 1. \tag{114}$$

Define

$$\tilde{J} = U^* J : \, W^r_p(Q) \to \widetilde{W}^s_{q^*}(Q)^*. \tag{115}$$

Recall that $e_n^{\rm ran}$ denotes the randomized n-th minimal error, the definition of which can be found in [11], section 2.

Theorem 4.2. *Let $r, s \in \mathbb{N}_0$, $1 \le p, q \le \infty$ and assume that (95) holds. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ with $n \ge 2$*

$$\begin{aligned} c_1 n^{-\gamma} &\le e_n^{\rm ran}(\tilde{J}, \mathcal{B}_{W^r_p(Q)}, \widetilde{W}^s_{q^*}(Q)^*) \\ &\le e_n^{\rm ran}(J, \mathcal{B}_{W^r_p(Q)}, W^s_{q^*}(Q)^*) \le c_2 n^{-\gamma} (\log n)^\nu, \end{aligned}$$

where

$$\gamma = \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d} + 1 - \frac{1}{\bar{p}}\right),$$

$$\nu = \begin{cases} \nu_1 & \text{if} \quad \gamma > 0, \\ 0 & \text{if} \quad \gamma = 0, \end{cases} \tag{116}$$

and ν_1 is given by (77).

Proof. If $\gamma = 0$, the upper bound follows from the boundedness of J. If $\gamma > 0$, Proposition 4.1 implies

$$e_{c_1 n}^{\rm ran}(J, \mathcal{B}_{W^r_p(Q)}, W^s_{q^*}(Q)^*) \le c_2 n^{-\gamma} (\log n)^{\nu_1}.$$

Monotonicity of the $e_n^{\rm ran}$ and an index shift yield the desired estimate. Then the result for $\tilde{J}$ follows from (114) and (115).

Now we show the lower bound. Because of (114) and (115), it suffices to consider $\tilde{J}$. We give four estimates, which together yield the needed result. Let $Q' = x_0' + [0, b']^d$ be a closed axis-parallel cube with $Q' \subset Q$ and let η be a C^∞ function with $\eta \equiv 1$ on Q' and $\operatorname{supp} \eta \subset Q$. Let $I_\eta : \widetilde{W}^s_{q^*}(Q)^* \to \mathbb{K}$ be the functional

$$I_\eta(f) = (f, \eta). \tag{117}$$

It follows that

$$\begin{aligned} & e_n^{\rm ran}(I_\eta \tilde{J}, \mathcal{B}_{W^r_p(Q)}, \mathbb{K}) \\ & \le \|I_\eta : \widetilde{W}^s_{q^*}(Q)^* \to \mathbb{K}\| \, e_n^{\rm ran}(\tilde{J}, \mathcal{B}_{W^r_p(Q)}, \widetilde{W}^s_{q^*}(Q)^*). \end{aligned} \tag{118}$$

Let ψ be a C^∞ function on $\mathbb{R}^d$ with support in the interior of $[0,1]^d$ and

$$\int_{[0,1]^d} \psi(x)dx \neq 0.$$

Let $n \in \mathbb{N}$, and put

$$k = \left\lceil \frac{\log n + 1}{d} \right\rceil,$$

hence

$$2^{d(k-1)} < 2n \leq 2^{dk}.$$

Put

$$\psi_i = R'_{ki}\psi \quad (1 \leq i \leq 2^{dk}),$$

where R'_{ki} is defined analogously to R_{ki} in (13), with $\tilde{Q}$ replaced by Q'. There are constants $c_1, c_2 > 0$ such that for all $(a_i) \in \mathbb{K}^{2^{dk}}$,

$$c_1\, 2^{rk-dk/p}\, \|(a_i)\|_{\ell_p^{2^{dk}}} \leq \Big\| \sum_{i=1}^{2^{dk}} a_i \psi_i \Big\|_{W_p^r(Q)} \leq c_2\, 2^{rk-dk/p}\, \|(a_i)\|_{\ell_p^{2^{dk}}}. \tag{119}$$

Consequently,

$$\begin{aligned} \Big\| \sum_{i=1}^{2^{dk}} a_i \tilde{J}\psi_i \Big\|_{\widetilde{W}_{q^*}^s(Q)^*} &\geq \sup_{\|(b_i)\|_{\ell_{q^*}^{2^{dk}}}=1} \left(\Big\| \sum_{i=1}^{2^{dk}} b_i\psi_i \Big\|_{\widetilde{W}_{q^*}^s(Q)}^{-1} \Big| \Big(\sum_{i=1}^{2^{dk}} a_i\psi_i, \sum_{i=1}^{2^{dk}} b_i\psi_i \Big) \Big| \right) \\ &\geq c\, 2^{-sk+dk/q^*-dk} \sup_{\|(b_i)\|_{\ell_{q^*}^{2^{dk}}}=1} \Big| \sum_{i=1}^{2^{dk}} a_i b_i \Big| \\ &= c\, 2^{-sk-dk/q} \|(a_i)\|_{\ell_q^{2^{dk}}}. \end{aligned} \tag{120}$$

Moreover, we have

$$|I_\eta \tilde{J}\psi_i| \geq c\, 2^{-dk}. \tag{121}$$

Now we consider the counting measure on

$$\left\{ \pm \|\psi_i\|_{W_p^r(Q)}^{-1} \psi_i \,:\, i = 1, \dots, 2^{dk} \right\},$$

use the relation of the randomized to the average minimal error (see [13, 17, 7]), and obtain from (119) and (120)

$$\begin{aligned} e_n^{\mathrm{ran}}(\tilde{J}, \mathcal{B}_{W_p^r(Q)}, \widetilde{W}_{q^*}^s(Q)^*) &\geq \frac{2^{dk}-n}{2^{dk}} \min_{1\leq i\leq 2^{dk}} \frac{\|\tilde{J}\psi_i\|_{\widetilde{W}_{q^*}^s(Q)^*}}{\|\psi_i\|_{W_p^r(Q)}} \\ &\geq c\, 2^{-(r+s)k+(1/p-1/q)dk} \\ &\geq cn^{-(r+s)/d+1/p-1/q}. \end{aligned} \tag{122}$$

Similarly, from (119) and (121)

$$\begin{aligned} e_n^{\mathrm{ran}}(I_\eta \tilde{J}, \mathcal{B}_{W_p^r(Q)}, \mathbb{K}) &\geq \frac{2^{dk}-n}{2^{dk}} \min_{1\leq i\leq 2^{dk}} \frac{|I_\eta \tilde{J}\psi_i|}{\|\psi_i\|_{W_p^r(Q)}} \\ &\geq c\, 2^{-dk-rk+dk/p} \geq cn^{-r/d-1+1/p}. \end{aligned} \tag{123}$$

For the other two estimates let ε_i ($i = 1, \dots, 2^{dk}$) be independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = 1/2$. Using again the average minimal error, this time with respect to the joint distribution of the ε_i, we get from (119) and (120)

$$\begin{aligned} & e_n^{\mathrm{ran}}(\tilde{J}, \mathcal{B}_{W_p^r(Q)}, \widetilde{W}_{q^*}^s(Q)^*) \\ &\geq \frac{\min\left\{\mathbb{E}\left\|\sum_{i\in\mathcal{I}} \varepsilon_i \tilde{J}\psi_i\right\|_{\widetilde{W}_{q^*}^s(Q)^*} : \mathcal{I}\subseteq\{1,\dots,2^{dk}\},\, |\mathcal{I}|\geq 2^{dk}-n\right\}}{\max\left\{\left\|\sum_{i=1}^{2^{dk}} a_i\psi_i\right\|_{W_p^r(Q)} : a_i\in\{-1,1\}, i=1,\dots,2^{dk}\right\}} \\ &\geq c\, 2^{-sk-rk}(2^{-dk}(2^{dk}-n))^{1/q} \geq cn^{-(r+s)/d}. \end{aligned} \tag{124}$$

By Khintchine's inequality, for any subset

$$\mathcal{I} \subseteq \{1, \dots, 2^{dk}\},$$

we have

$$\mathbb{E}\left|\sum_{i\in\mathcal{I}} \varepsilon_i I_\eta \tilde{J}\psi_i\right| \geq c\left(\mathbb{E}\left|\sum_{i\in\mathcal{I}} \varepsilon_i I_\eta \tilde{J}\psi_i\right|^2\right)^{1/2} \geq c\, 2^{-dk}|\mathcal{I}|^{1/2}. \tag{125}$$

From (119) and (125) we obtain

$$\begin{aligned} & e_n^{\mathrm{ran}}(I_\eta \tilde{J}, \mathcal{B}_{W_p^r(Q)}, \mathbb{K}) \\ &\geq \frac{\min\left\{\mathbb{E}\left|\sum_{i\in\mathcal{I}} \varepsilon_i I_\eta \tilde{J}\psi_i\right| : \mathcal{I}\subseteq\{1,\dots,2^{dk}\},\, |\mathcal{I}|\geq 2^{dk}-n\right\}}{\max\left\{\left\|\sum_{i=1}^{2^{dk}} a_i\psi_i\right\|_{W_p^r(Q)} : a_i\in\{-1,1\}, i=1,\dots,2^{dk}\right\}} \\ &\geq c\, 2^{-dk-rk}(2^{dk}-n)^{1/2} \geq cn^{-r/d-1/2}. \end{aligned} \tag{126}$$

Now (118), (122), (123), (124), and (126) together imply the lower bound in Theorem 4.2. □

We note that the same lower bound techniques also apply to the larger class of randomized adaptive nonlinear algorithms (as described, e.g., in [8, 9]) and thus Theorem 4.2 also holds for the n-th minimal error with respect to this class.

By definition, see [1], section 3.11, for $1 < q \le \infty$ and $s > 0$

$$\widetilde{W}_{q^*}^s(Q)^* = W_q^{-s}(Q). \tag{127}$$

Clearly, (127) also holds for $s = 0$.

Corollary 4.3. *Let $1 < q \le \infty$. With the assumptions and notation of Theorem 4.2 we have*

$$c_1 n^{-\gamma} \le e_n^{\rm ran}(\tilde{J}, \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q)) \le c_2 n^{-\gamma}(\log n)^\nu.$$

5 Deterministic setting

Let $r \in \mathbb{N}_0$. Then $W_p^r(Q)$ is continuously embedded into $C(\bar{Q})$, with $\bar{Q}$ the closure of Q, if and only if

$$\left.\begin{array}{lll} & p = 1 & \text{and} \quad r/d \ge 1 \\ \text{or} & & \\ & 1 < p \le \infty & \text{and} \quad r/d > 1/p \end{array}\right\} \tag{128}$$

see [1]. In these cases function values are well-defined. Consequently, deterministic algorithms based on them make sense.

The following theorem is the analogue of Theorem 4.2 for the deterministic setting. Most of it is known. Respective estimates for Besov and Triebel-Lizorkin spaces can be found in Vybíral [21], which, in turn, are based on results of Novak and Triebel [14]. The case of $\tilde{J}$ of the theorem below follows from [21] (taking into account also the relations between Sobolev and Besov spaces, see [20]), with the exception of the case $s/d = 1/p - 1/q$ for $1 \le p < q \le \infty$, which was left open in [21].

Below we settle this case up to a logarithmic factor. Parts of it still follow by the same method as used in [21], however, the subcase described by relation (105) of the present paper requires a somewhat different approach. This is the new part of the following result. For completeness, the short proof of the other cases is included.

The numbers $e_n^{\rm det}$ stand for the deterministic n-th minimal error (see [11]).

Theorem 5.1. *Let $r, s \in \mathbb{N}_0$, $1 \le p, q \le \infty$ and assume that (95) and (128) hold. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ with $n \ge 2$*

$$\begin{aligned} c_1 n^{-\gamma'} &\le e_n^{\rm det}(\tilde{J}, \mathcal{B}_{W_p^r(Q)}, \widetilde{W}_{q^*}^s(Q)^*) \\ &\le e_n^{\rm det}(J, \mathcal{B}_{W_p^r(Q)}, W_{q^*}^s(Q)^*) \le c_2 n^{-\gamma'}(\log n)^{\nu'}, \end{aligned}$$

where

$$\begin{aligned}\gamma' &= \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d}\right), \\ \nu' &= \begin{cases} 1 & \textit{if} \quad \frac{s}{d} = \frac{1}{q^*},\ p = 1,\ 1 < q < \infty, \\ 0 & \textit{otherwise.} \end{cases}\end{aligned}$$

Proof. We use the factorization and consider the same three cases as in the proof of Proposition 4.1:

$$J : W_p^r(Q) \xrightarrow{J_0} L_w(Q) \xrightarrow{J_1} W_{q^*}^s(Q)^*.$$

Let $n \in \mathbb{N}$, $n \geq 2$, put

$$L = \max\left(\left\lceil \frac{\log n}{d} \right\rceil, l_0\right), \tag{129}$$

and let $P_{L,0}$ be the operator described in Proposition 4.1 of [11].

In case 1, that is, if (8) holds, we have $w = p$. Then we conclude from Proposition 4.1 of [11]

$$\begin{aligned}&\sup_{f \in \mathcal{B}_{W_p^r(Q)}} \|Jf - J_1 P_{L,0} f\|_{W_{q^*}^s(Q)^*} \\ &\leq \|J_1\| \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \|f - P_{L,0} f\|_{L_p(Q)} \leq c\, 2^{-rL} \leq cn^{-r/d}.\end{aligned}$$

In case 3, meaning that (106) holds, we have by (110)

$$\frac{1}{w} = \frac{s}{d} + \frac{1}{q} < \frac{1}{p},$$

hence, as shown there, (92) is satisfied for the pair (p, w) and (8) for the pair (w, q). We get, using again Proposition 4.1 of [11],

$$\begin{aligned}&\sup_{f \in \mathcal{B}_{W_p^r(Q)}} \|Jf - J_1 P_{L,0} f\|_{W_{q^*}^s(Q)^*} \\ &\leq \|J_1\| \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \|f - P_{L,0} f\|_{L_w(Q)} \leq c\, 2^{-rL + (1/p - 1/w)dL} \\ &\leq cn^{-r/d + 1/p - 1/w} = cn^{-(r+s)/d + 1/p - 1/q}.\end{aligned}$$

It remains to consider case 2,

$$\frac{s}{d} = \frac{1}{q^*}, \quad p = 1, \quad 1 < q < \infty. \tag{130}$$

Here w was chosen in such a way that (107) holds. Again, (92) is fulfilled for (p,w) and (8) for (w,q). Let $f \in \mathcal{B}_{W_1^r(Q)}$. Then we have, with $I : L_w(Q) \to L_{w^*}(Q)^*$ from (6) and $P_L : W_{q^*}^s(Q) \to L_{w^*}(Q)$ as defined in (19),

$$\begin{aligned} &\|Jf - J_1 P_{L,0} f\|_{W_{q^*}^s(Q)^*} \\ &= \|J_1(J_0 f - P_{L,0} f)\|_{W_{q^*}^s(Q)^*} \\ &\le \|(J_1 - P_L^* I)(J_0 f - P_{L,0} f)\|_{W_{q^*}^s(Q)^*} + \|P_L^* I(J_0 f - P_{L,0} f)\|_{W_{q^*}^s(Q)^*}. \end{aligned} \tag{131}$$

Taking into account (5), using Lemma 3.2 above and Proposition 4.1 of [11], we estimate the first summand as

$$\begin{aligned} &\|(J_1 - P_L^* I)(J_0 f - P_{L,0} f)\|_{W_{q^*}^s(Q)^*} \\ &\le \|(J_{1,0}^* I - P_L^* I)(J_0 f - P_{L,0} f)\|_{W_{q^*}^s(Q)^*} \\ &\le \|J_{1,0}^* I - P_L^* I : L_w(Q) \to W_{q^*}^s(Q)^*\| \|f - P_{L,0} f\|_{L_w(Q)} \\ &\le \|J_{1,0} - P_L : W_{q^*}^s(Q) \to L_{w^*}(Q)\| \|f - P_{L,0} f\|_{L_w(Q)} \\ &\le c\, 2^{-sL + (1/w - 1/q)dL - rL + (1 - 1/w)dL} = c\, 2^{-rL} \le c n^{-r/d}, \end{aligned} \tag{132}$$

the equality in the last line being a consequence of (130). The second summand in (131) is treated as follows. We consider the involved operators acting as

$$P_L^* I(J_0 - P_{L,0}) : W_1^r(Q) \xrightarrow{J_0 - P_{L,0}} L_1(Q) \xrightarrow{P_L^* I} W_{q^*}^s(Q)^*.$$

Then we get, using Proposition 4.1 of [11] again,

$$\begin{aligned} \|P_L^* I(J_0 f - P_{L,0} f)\|_{W_{q^*}^s(Q)^*} &\le \|P_L^* I : L_1(Q) \to W_{q^*}^s(Q)^*\| \|f - P_{L,0} f\|_{L_1(Q)} \\ &\le c\, 2^{-Lr} \|P_L^* I : L_1(Q) \to W_{q^*}^s(Q)^*\| \\ &\le c n^{-r/d} \|P_L : W_{q^*}^s(Q) \to L_\infty(Q)\|. \end{aligned} \tag{133}$$

We have by (31),

$$\begin{aligned} \|P_L : W_{q^*}^s(Q) \to L_\infty(Q)\| &\le \sum_{l=l_0}^{L} \|T_l : W_{q^*}^s(Q) \to L_\infty(Q)\| \\ &\le \sum_{l=l_0}^{L} \|\tilde{T}_l : W_{q^*}^s(Q) \to L_\infty(Q_l)\|, \end{aligned} \tag{134}$$

with $\tilde{T}_l$ and T_l defined in (29) and (30). By Lemma 3.3, for $g \in W_{q^*}^s(Q)$,

$$\tilde{T}_l g \in \operatorname{span}\{\chi_{Q_{lk}} R_{lk} \varphi_j \ :\ k \in \mathcal{K}_l,\, 1 \le j \le \kappa\}.$$

From (33) and (34) we conclude that for any $b_{kj} \in \mathbb{K}$ ($k \in \mathcal{K}_l, j = 1, \dots, \kappa$),

$$\begin{aligned} \Big\| \sum_{k \in \mathcal{K}_l} \sum_{j=1}^{\kappa} b_{kj} \chi_{Q_{lk}} R_{lk} \varphi_j \Big\|_{L_\infty(Q_l)} &\le c \|(b_{kj})\|_{\ell_\infty^{n_l}} \le c \|(b_{kj})\|_{\ell_{w^*}^{n_l}} \\ &\le c\, 2^{dl/w^*} \Big\| \sum_{k \in \mathcal{K}_l} \sum_{j=1}^{\kappa} b_{kj} \chi_{Q_{lk}} R_{lk} \varphi_j \Big\|_{L_{w^*}(Q_l)}. \end{aligned}$$

Consequently, using also (37) of Lemma 3.3 and (130), we get

$$\begin{aligned}\|\tilde{T}_l : W^s_{q^*}(Q) \to L_\infty(Q_l)\| &\leq c\,2^{dl/w^*}\|\tilde{T}_l : W^s_{q^*}(Q) \to L_{w^*}(Q_l)\| \\ &\leq c\,2^{dl/w^*-sl+(1/w-1/q)dl} = c. \end{aligned} \tag{135}$$

Combining (133), (134), and (135), and using (129), we obtain

$$\|P^*_L I(J_0 f - P_{L,0}f)\|_{W^s_{q^*}(Q)^*} \leq cn^{-r/d}\log n,$$

and with (131) and (132) we arrive at

$$\|Jf - J_1 P_{L,0}f\|_{W^s_{q^*}(Q)^*} \leq cn^{-r/d}\log n,$$

which proves the upper bound also in case 2.

The lower bound follows by standard techniques from information-based complexity [17], Ch. 3.1, using relations (119), (120), the analogue of (118) for the deterministic case, and (121).

□

Let us compare the results for the deterministic and the randomized setting. In the table below we present the order of the n-th minimal error of $\tilde{J} : W^r_p(Q) \to W^{-s}_q(Q)$ up to logarithmic factors, for $r, s \in \mathbb{N}_0$, $1 \leq p, q \leq \infty$ satisfying (95) and (128) (with the convention that for $q = 1$ one has to replace $W^{-s}_1(Q)$ by $\widetilde{W}^s_\infty(Q)^*$).

$J : W^r_p(Q) \to W^{-s}_q(Q)$	e^{det}_n	e^{ran}_n
$1 \leq q \leq p \leq \infty$,	$n^{-\frac{r}{d}}$	$n^{-\frac{r}{d}-\min\left(\frac{s}{d},1-\frac{1}{p}\right)}$
$1 \leq p < q \leq \infty$, $\frac{s}{d} > \frac{1}{p} - \frac{1}{q}$	$n^{-\frac{r}{d}}$	$n^{-\frac{r}{d}-\min\left(\frac{s}{d}-\frac{1}{p}+\frac{1}{q},1-\frac{1}{p}\right)}$
$1 \leq p < q \leq \infty$, $\frac{s}{d} \leq \frac{1}{p} - \frac{1}{q}$	$n^{-\frac{r+s}{d}+\frac{1}{p}-\frac{1}{q}}$	$n^{-\frac{r+s}{d}+\frac{1}{p}-\frac{1}{q}}$

In the first two cases there is a speedup of randomized algorithms over deterministic ones, as soon as $s > 0$, $p > 1$, and it can reach the magnitude $n^{-1/2}$. In the third case there is no speedup.

The case that condition (128) of embedding into $C(\bar{Q})$ does not hold, is also of interest. Here values of $W^r_p(Q)$ functions are not well-defined, and thus, neither is e^{det}_n. Therefore, we restrict our considerations to the dense subset $\mathcal{B}_{W^r_p(Q)} \cap C(\bar{Q})$ of $\mathcal{B}_{W^r_p(Q)}$, on which function values are correctly defined. It turns out though that deterministic algorithms can give no non-trivial convergence rate at all, as the following result shows. It is an extension of Proposition 2 of [10] ($s = 0$) and complements Theorem 4.3 of [11] ($s \geq 0$).

Theorem 5.2. *Let $r,s \in \mathbb{N}_0$, $1 \le p,q \le \infty$ and assume that (95) holds, but (128) does not. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$*

$$\begin{aligned} c_1 &\le e_n^{\rm det}(\tilde{J}, \mathcal{B}_{W_p^r(Q)} \cap C(\bar{Q}), \widetilde{W}_{q^*}^s(Q)^*) \\ &\le e_n^{\rm det}(J, \mathcal{B}_{W_p^r(Q)} \cap C(\bar{Q}), W_{q^*}^s(Q)^*) \le c_2. \end{aligned} \tag{136}$$

Proof. The upper bound follows from the boundedness of J. Let us turn to the lower bound. Observe that (128) does not hold iff

$$p = 1 \quad \text{and} \quad r/d < 1 \tag{137}$$

or

$$1 < p < \infty \quad \text{and} \quad r/d \le 1/p \tag{138}$$

or

$$p = \infty \quad \text{and} \quad r = 0. \tag{139}$$

It was shown in [10], Lemma 1, that if (137) or (138) hold, then there exists a sequence of functions

$$(f_m)_{m=1}^\infty \subset W_p^r(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \tag{140}$$

such that for all m

$$f_m(0) = 1, \quad \operatorname{supp} f_m \subseteq B\left(0, \frac{1}{m}\right), \tag{141}$$

and

$$\lim_{m\to\infty} \|f_m\|_{W_p^r(\mathbb{R}^d)} = 0.$$

If (139) holds, it is readily seen that there is a sequence satisfying (140), (141), and the following condition

$$0 \le f_m(x) \le 1 \quad (x \in \mathbb{R}^d,\, m \in \mathbb{N}).$$

Now we combine the proof of Proposition 2 of [10] with that of Theorem 4.2 above. Let η and I_η be as defined there, see (117). Here we assume that η satisfies, in addition,

$$\int_Q \eta(x)dx > 0.$$

In analogy to (118) we have

$$\begin{aligned} & e_n^{\rm det}(I_\eta \tilde{J}, \mathcal{B}_{W_p^r(Q)} \cap C(\bar{Q}), \mathbb{K}) \\ &\quad \le \|I_\eta : \widetilde{W}_{q^*}^s(Q)^* \to \mathbb{K}\| \, e_n^{\rm det}(\tilde{J}, \mathcal{B}_{W_p^r(Q)} \cap C(\bar{Q}), \widetilde{W}_{q^*}^s(Q)^*). \end{aligned} \tag{142}$$

Now fix any distinct points $x_1, \dots, x_n \in Q$. For $m \in \mathbb{N}$ define $g_m \in C(\bar{Q})$ by

$$g_m(x) = 1 - \sum_{i=1}^n f_m(x - x_i) \quad (x \in \bar{Q}).$$

Then

$$\lim_{m\to\infty} \|g_m\|_{W_p^r(Q)} = 1.$$

Furthermore,

$$\begin{aligned} I_\eta \widetilde{J} g_m &= \int_Q \eta(x) g_m(x) dx = \int_Q \eta(x) dx - \sum_{i=1}^n \int_Q \eta(x) f_m(x - x_i) dx \\ &\geq \int_Q \eta(x) dx - n \|\eta\|_{C(\bar{Q})} \|f_m\|_{L_1(\mathbb{R}^d)} \to \int_Q \eta(x) dx \end{aligned}$$

as $m \to \infty$. Moreover, (141) implies that

$$g_m(x_i) = 0 \quad (i = 1, \dots, n)$$

for m sufficiently large. An application of standard lower bound results, see [17], Ch. 3.1, gives

$$e_n^{\rm det}(I_\eta \tilde{J}, \mathcal{B}_{W_p^r(Q)} \cap C(\bar{Q}), \mathbb{K}) \geq \int_Q \eta(x) dx > 0,$$

which together with (142) shows the lower bound of (136) and concludes the proof.

□

Now we can again compare with the randomized setting, with $r, s \in \mathbb{N}_0$, $1 \leq p, q \leq \infty$ satisfying (95), omitting logarithmic factors:

$J : W_p^r(Q) \to W_q^{-s}(Q)$	$e_n^{\rm det}$	$e_n^{\rm ran}$
$p = 1$ and $\frac{r}{d} < 1$ or $1 < p \leq \infty$ and $\frac{r}{d} \leq \frac{1}{p}$	1	$n^{-\min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d} + 1 - \frac{1}{p}\right)}$

So here the speedup can be as much as n^{-1}, which is the case if $r/d = 1/p$, $1 < p \leq 2$, and $s/d \geq 1 - 1/\max(p, q)$.

6 Other function spaces

Here we extend the results to Besov spaces $B^r_{pu}(Q)$ for $r \in \mathbb{R}$, $r > 0$, $1 \le p, u \le \infty$, and Bessel potential spaces $H^r_p(Q)$ for $r \in \mathbb{R}$, $r > 0$, $1 < p < \infty$. For notation and related facts we refer to section 5 of [11] and the references given there. Let

$$\mathcal{E} : W^r_p(Q) \to W^r_p(\mathbb{R}^d) \quad (r \in \mathbb{N}_0,\ 1 \le p \le \infty)$$

be a universal extension operator (see [16], Ch. VI, Th. 5). It follows by interpolation that $\mathcal{E}$ is also an extension operator for the spaces B^r_{pu} $(r > 0,\ 1 \le p, u \le \infty)$ and H^r_p $(r \ge 0,\ 1 < p < \infty)$, see also section 2.4 of [19]. First we state an analogue of Lemma 3.2.

Lemma 6.1. *Let*

$$1 \le p, q, v \le \infty \tag{143}$$

in the case of Besov spaces, and

$$1 < p, q < \infty \tag{144}$$

in the case of Bessel potential spaces. Let $s \in \mathbb{R}$ and assume

$$\frac{s}{d} > \left(\frac{1}{p} - \frac{1}{q}\right)_+ . \tag{145}$$

Let $\tilde{P}_l$ for $l \in \mathbb{N}_0$, $l \ge l_0$ be given by (18). Then there is a constant $c > 0$ such that for all $l \in \mathbb{N}_0$, $l \ge l_0$

$$\sup_{f \in \mathcal{B}_{B^s_{q^* v^*}(Q)}} \|\mathcal{E}f - \tilde{P}_l f\|_{L_{p^*}(Q_l)} \le c\, 2^{-sl + \max(1/p - 1/q, 0)dl}$$

and

$$\sup_{f \in \mathcal{B}_{H^s_{q^*}(Q)}} \|\mathcal{E}f - \tilde{P}_l f\|_{L_{p^*}(Q_l)} \le c\, 2^{-sl + \max(1/p - 1/q, 0)dl}.$$

Proof. The proof is similar to that of Proposition 5.1 of [11]. We only show the case of Besov spaces, the case of Bessel potential spaces follows analogously, just using complex interpolation.

Consider first the case $p = q$. We put $s_0 = \lceil s \rceil - 1$ and $s_1 = \lfloor s \rfloor + 1$. Let $0 < \vartheta < 1$ be such that $s = (1 - \vartheta)s_0 + \vartheta s_1$. By Lemma 3.2

$$\sup_{f \in \mathcal{B}_{W^{s_i}_{p^*}(Q)}} \|\mathcal{E}f - \tilde{P}_l f\|_{L_{p^*}(Q_l)} \le c\, 2^{-s_i l} \quad (i = 0, 1).$$

Using real interpolation we get

$$\sup_{f \in \mathcal{B}_{B^s_{p^* v^*}(Q)}} \|\mathcal{E}f - \tilde{P}_l f\|_{L_{p^*}(Q_l)} \le c\, 2^{-sl}.$$

For $p \neq q$ we put

$$s_1 = s - d\left(\frac{1}{p} - \frac{1}{q}\right)_+ = s - d\left(\frac{1}{q^*} - \frac{1}{p^*}\right)_+ .$$

Then $s_1 > 0$ and the embedding $B^s_{q^*v^*}(Q) \to B^{s_1}_{p^*v^*}(Q)$ is continuous (see the references in the proof of Proposition 5.1 of [11]). It follows that

$$\begin{aligned} \sup_{f \in \mathcal{B}_{B^s_{q^*v^*}(Q)}} \|\mathcal{E}f - \tilde{P}_l f\|_{L_{p^*}(Q_l)} &\leq c \sup_{f \in \mathcal{B}_{B^{s_1}_{p^*v^*}(Q)}} \|\mathcal{E}f - \tilde{P}_l f\|_{L_{p^*}(Q_l)} \\ &\leq c\, 2^{-s_1 l} = c\, 2^{-sl + \max(1/p - 1/q, 0)dl}. \end{aligned}$$

□

Let

$$J_1^{\mathrm{B}} : L_p(Q) \to B^s_{q^*v^*}(Q)^*, \quad J_1^{\mathrm{H}} : L_p(Q) \to H^s_{q^*}(Q)^*$$

be the embeddings defined analogously to (4), (5), (6), and (7). If (145) holds, they are well-defined and continuous. Put

$$\nu_0^{\mathrm{B}} = \nu_0^{\mathrm{B}}(p, q, v) = \begin{cases} \min(p, q, v, 2) & \text{if} \quad q < \infty \\ 1 & \text{if} \quad q = \infty \end{cases} \tag{146}$$

and

$$\nu_1^{\mathrm{B}} = \nu_1^{\mathrm{B}}(s, p, q, v) =$$

$$\begin{cases} 0 & \text{if} \quad \theta > \tau \\ 1 - \dfrac{1}{\min(p,2)} + \dfrac{1}{\min(p,v,2)} & \text{if} \quad \theta = \tau \quad \text{and} \quad p \leq q < \infty \\ 2 - \dfrac{1}{\min(p,2)} & \text{if} \quad \theta = \tau \quad \text{and} \quad p < q = \infty \\ 2 & \text{if} \quad \theta = \tau \quad \text{and} \quad p = q = \infty \\ 1 & \text{if} \quad \theta = \tau \quad \text{and} \quad p > q \\ 0 & \text{if} \quad \theta < \tau \quad \text{and} \quad \min(p,q) < \infty \\ \theta & \text{if} \quad \theta < \tau \quad \text{and} \quad p = q = \infty, \end{cases} \tag{147}$$

where θ and τ are defined in (76). The counterpart of Proposition 3.6 reads as follows.

Proposition 6.2. *Let Q be a bounded Lipschitz domain and assume that s, p, q, v satisfy (143–145). Let $p_1 < \infty$ be such that $1 \leq p_1 \leq p$. Then there are constants $c_1 \in \mathbb{N}$, $c_2 > 0$ such that for each $n \in \mathbb{N}$ with $n \geq 2$ there is a choice of parameters*

$L, (N_l)_{l=l_0}^L$ such that algorithm $A^{(1)}$ defined by (50), (53–55) satisfies in the Besov case $A^{(1)} \in \mathcal{A}^{\rm ran}_{c_1 n}(L_p(Q), B^s_{q^* v^*}(Q)^*)$ and

$$\sup_{g\in\mathcal{B}_{L_p(Q)}} \left(\mathbb{E}\,\|J_1^{\rm B} g - A^{(1)}_\omega(g)\|^{p_1}_{B^s_{q^*v^*}(Q)^*}\right)^{1/p_1} \le c_2 n^{-\min(\theta,\tau)}(\log n)^{\nu_1^{\rm B}},$$

and in the Bessel potential case $A^{(1)} \in \mathcal{A}^{\rm ran}_{c_1 n}(L_p(Q), H^s_{q^*}(Q)^*)$ and

$$\sup_{g\in\mathcal{B}_{L_p(Q)}} \left(\mathbb{E}\,\|J_1^{\rm H} g - A^{(1)}_\omega(g)\|^{p_1}_{H^s_{q^*}(Q)^*}\right)^{1/p_1} \le c_2 n^{-\min(\theta,\tau)}(\log n)^{\nu_1},$$

with θ and τ given by (76), $\nu_1^{\rm B}$ by (147), and ν_1 by (77).

Proof. With Lemma 6.1 at hand, the counterparts of Lemmas 3.4 and 3.5 and, based on them, Proposition 6.2 can be proved in literally the same way, with just some minor modifications, which we shortly discuss here.

In Lemmas 3.4 and 3.5 the assumption of (8) has to be replaced by (143–145). Moreover, in the case of Besov spaces the parameter ν_0 has to be replaced by $\nu_0^{\rm B}$ from (146).

In the proof for Besov spaces, we can assume without loss of generality that $v \le 2$, since the statements for $v > 2$ follows from the case $v = 2$. To see this, note first that for $v > 2$ we have $\nu_0(p,q,v) = \nu_0(p,q,2)$ and $\nu_1^{\rm B}(s,p,q,v) = \nu_1^{\rm B}(s,p,q,2)$. Now the estimates for $v > 2$ can be derived from those for $v = 2$ using the factorization

$$J_1^{\rm B} : L_p(Q) \xrightarrow{J_2} B^s_{q^*,2}(Q)^* \xrightarrow{J_3} B^s_{q^*v^*}(Q)^*,$$

where the continuity of J_2 is a consequence of (145) and J_3 is the adjoint of the continuous embedding

$$B^s_{q^*,v^*}(Q) \to B^s_{q^*,2}(Q),$$

see [18], Prop. 2.3.2.2.

We show that for $v \le 2$ the space $B^s_{q^*v^*}(Q)^*$ is of type $\nu_0^{\rm B}$. Since type 1 is trivial, we only have to consider the case $\nu_0^{\rm B} > 1$. This implies $1 < q < \infty$ and $v > 1$. The space $B^s_{q^*v^*}(Q)$ is isomorphic to a subspace of $\ell_{v^*}(L_{q^*}(\mathbb{R}^d))$, which follows from the respective extension theorem and from the definition of $B^s_{q^*v^*}(\mathbb{R}^d)$ (see [18], 2.3.1, Definition 2(i)). Hence $B^s_{q^*v^*}(Q)^*$ is isomorphic to a quotient of $\ell_v(L_q(\mathbb{R}^d))$. The space $\ell_v(L_q(\mathbb{R}^d))$ is of type $\min(q,v,2)$, and so is any quotient (see [12], p. 247). Thus, $B^s_{q^*v^*}(Q)^*$ is also of type $\nu_0^{\rm B} \le \min(q,v,2)$.

In the case of Bessel potential spaces we observe that by the corresponding extension theorem, $H^s_{q^*}(Q)$ is isomorphic to a subspace of $H^s_{q^*}(\mathbb{R}^d)$. By definition, the latter space can be identified with a subspace of $L_{q^*}(\mathbb{R}^d)$ (see [18], 2.2.2, relation (11)). Then we argue as in the proof of Lemma 3.5 to conclude that $H^s_{q^*}(Q)^*$ is of type ν_0, with ν_0 from (67).

For Besov spaces we also have to modify the factorization (88) as follows:

$$J_1^{\mathrm{B}} : L_p(Q) \xrightarrow{J_{1,1}} L_{p_1}(Q) \xrightarrow{J_{1,2}} B^s_{p_1^*,2}(Q)^* \xrightarrow{J_{1,3}} B^s_{q^*v^*}(Q)^*.$$

(see again [18], Prop. 2.3.2.2).

□

Let

$$1 \le p, q, u, v \le \infty \tag{148}$$

in the case of Besov spaces, and

$$1 < p, q < \infty \tag{149}$$

in the case of Bessel potential spaces. Let $r, s \in \mathbb{R}$ be such that

$$r, s > 0, \qquad \frac{r+s}{d} > \left(\frac{1}{p} - \frac{1}{q}\right)_+ . \tag{150}$$

We introduce the embeddings

$$J^{\mathrm{B}} : B^r_{pu}(Q) \to B^s_{q^*v^*}(Q)^*, \quad J^{\mathrm{H}} : H^r_p(Q) \to H^s_{q^*}(Q)^*$$

by analogy with (90–91). Then J^{B} and J^{H} are well-defined and continuous. This is easy to show directly and also follows from the proof of Proposition 6.3 below. Consider the condition

$$\left.\begin{array}{l} p = 1,\ \dfrac{s}{d} + \dfrac{1}{q} = 1,\ v = 1 \\ \text{or} \\ p = 1,\ \dfrac{s}{d} = 1,\ q = \infty. \end{array}\right\} \tag{151}$$

We define for $\delta > 0$

$$\nu_2^{\mathrm{B}}(\delta) = \nu_2^{\mathrm{B}}(\delta, s, p, q, v) = \begin{cases} 1 + \delta & \text{if (151) holds} \\ \nu_1^{\mathrm{B}}(s, p, q, v) & \text{otherwise,} \end{cases} \tag{152}$$

with $\nu_1^{\mathrm{B}}(s, p, q, v)$ given by (147).

Proposition 6.3. *Assume r, s, p, q, u, v satisfy (148–150), let $p_1 < \infty$ be such that $1 \le p_1 \le p$, and let $\delta > 0$. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ with $n \ge 2$ there is a choice of parameters k, L, $(N_l)_{l=l_0}^L$ such that algorithm A defined in (96) satisfies in the case of Besov spaces $A \in \mathcal{A}^{\mathrm{ran}}_{c_1 n}(B^r_{pu}(Q), B^s_{q^*v^*}(Q)^*)$ and*

$$\sup_{f \in \mathcal{B}_{B^r_{pu}(Q)}} \left(\mathbb{E}\, \|J^{\mathrm{B}} f - A_\omega(f)\|^{p_1}_{B^s_{q^*v^*}(Q)^*}\right)^{1/p_1} \le c_2 n^{-\gamma} (\log n)^{\nu_2^{\mathrm{B}}(\delta)},$$

and in the case of Bessel potential spaces, $A \in \mathcal{A}^{\mathrm{ran}}_{c_1 n}(H^r_p(Q), H^s_{q^*}(Q)^*)$ *and*

$$\sup_{f\in\mathcal{B}_{H^r_p(Q)}} \left(\mathbb{E}\,\|J^{\mathrm{H}} f - A_\omega(f)\|^{p_1}_{H^s_{q^*}(Q)^*}\right)^{1/p_1} \le c_2 n^{-\gamma}(\log n)^{\nu_1},$$

where

$$\gamma = \min\left(\frac{r+s}{d} - \left(\frac{1}{p}-\frac{1}{q}\right)_+, \frac{r}{d}+1-\frac{1}{\bar{p}}\right),$$

and $\nu_2^{\mathrm{B}}(\delta)$ *and* ν_1 *are defined in (152) and (77), respectively.*

Proof. We give the proof for the Besov case. The proof for Bessel potential spaces is analogous, just easier, since case 3.2 is excluded by (149).

We use the factorization of J^{B} as

$$J^{\mathrm{B}} : B^r_{pu}(Q) \xrightarrow{J_0^{\mathrm{B}}} L_w(Q) \xrightarrow{J_1^{\mathrm{B}}} B^s_{q^*v^*}(Q)^*,$$

with suitably chosen $1 \le w \le \infty$ satisfying

$$\frac{r}{d} > \left(\frac{1}{p}-\frac{1}{w}\right)_+, \quad \frac{s}{d} > \left(\frac{1}{w}-\frac{1}{q}\right)_+. \tag{153}$$

Hence, both embeddings J_0^{B} and J_1^{B} are continuous.

For $n \in \mathbb{N}$, $n \ge 2$, let the parameter k of algorithm A be chosen as in (100) of the proof of Proposition 4.1 and the parameters $L, (N_l)_{l=l_0}^L$ according to Proposition 6.2, with index pair (w, q). Then we have

$$A \in \mathcal{A}^{\mathrm{ran}}_{cn}(B^r_{pu}(Q), B^s_{q^*v^*}(Q)^*)$$

and, similarly to (102), for $1 \le t < \infty$

$$\begin{aligned}
&\sup_{f\in\mathcal{B}_{B^r_{pu}(Q)}} \left(\mathbb{E}\,\|J^{\mathrm{B}} f - A_\omega(f)\|^t_{B^s_{q^*v^*}(Q)^*}\right)^{1/t} \\
&\le \sup_{g\in\mathcal{B}_{L_w(Q)}} \left(\mathbb{E}_{\,\omega_1}\|J_1^{\mathrm{B}} g - A^{(1)}_{\omega_1}(g)\|^t_{B^s_{q^*v^*}(Q)^*}\right)^{1/t} \\
&\qquad \times \sup_{f\in\mathcal{B}_{B^r_{pu}(Q)}} \left(\mathbb{E}_{\,\omega_0}\left\|J_0^{\mathrm{B}} f - P^{(0)}_{k,\omega_0} f\right\|^t_{L_w(Q)}\right)^{1/t}.
\end{aligned} \tag{154}$$

The cases considered here are somewhat different from those in the proof of Proposition 4.1.

Case 1. Assume that (145) holds. Then we set $w = p$, use relation (55) of Proposition 5.1 in [11] to get

$$\sup_{f\in\mathcal{B}_{B^r_{pu}(Q)}} \left(\mathbb{E}_{\,\omega_0}\left\|J_0^{\mathrm{B}} f - P^{(0)}_{k,\omega_0} f\right\|^{p_1}_{L_p(Q)}\right)^{1/p_1} \le c n^{-r/d}, \tag{155}$$

and Proposition 6.2 above to obtain

$$\sup_{g\in\mathcal{B}_{L_p(Q)}} \left(\mathbb{E}_{\omega_1}\|J_1^{\mathrm{B}}g - A_{\omega_1}^{(1)}(g)\|_{B^s_{q^*v^*}(Q)^*}^{p_1}\right)^{1/p_1}$$
$$\leq \ cn^{-\min\left(\frac{s}{d}-\left(\frac{1}{p}-\frac{1}{q}\right)_+,1-\frac{1}{\bar{p}}\right)}(\log n)^{\nu_1^{\mathrm{B}}}. \tag{156}$$

Moreover, (145) excludes (151), so $\nu_1^{\mathrm{B}} = \nu_2^{\mathrm{B}}(\delta)$. Now the result follows from (154), (155), and (156).

In the rest of the proof we assume that (145) does not hold, that is,

$$\frac{s}{d} \leq \left(\frac{1}{p} - \frac{1}{q}\right)_+ .$$

Because of $s > 0$ this means that $p < q$, and hence

$$\frac{s}{d} + \frac{1}{q} \leq \frac{1}{p}. \tag{157}$$

It follows from (150) that

$$\max\left(\frac{1}{p} - \frac{r}{d}, \frac{1}{q}\right) < \frac{s}{d} + \frac{1}{q}.$$

In each of the following cases we choose w in such a way that

$$\max\left(\frac{1}{p} - \frac{r}{d}, \frac{1}{q}\right) < \frac{1}{w} < \frac{s}{d} + \frac{1}{q}. \tag{158}$$

From (157) and (158) we conclude that

$$p < w < q, \tag{159}$$

and consequently,

$$p_1 < w. \tag{160}$$

Moreover, (158) and (159) imply that (153) is satisfied. Now we use again (55) of Proposition 5.1 from [11], which yields

$$\sup_{f\in\mathcal{B}_{B^r_{pu}(Q)}} \left(\mathbb{E}_{\omega_0}\left\|J_0^{\mathrm{B}}f - P_{k,\omega_0}^{(0)}f\right\|_{L_w(Q)}^w\right)^{1/w} \leq cn^{-r/d+1/p-1/w}. \tag{161}$$

Case 2: We assume, in addition to (157), that

$$\frac{s}{d} + \frac{1}{q} \leq \frac{1}{2}. \tag{162}$$

It follows from (158) and (162) that $2 < w < \infty$ and

$$\begin{aligned}\theta(s,w,q) &= \frac{s}{d}-\left(\frac{1}{w}-\frac{1}{q}\right)_+ = \frac{s}{d}-\frac{1}{w}+\frac{1}{q}\\ &< \frac{1}{2}=1-\frac{1}{\bar{w}}=\tau(w),\end{aligned}$$

so (147) gives $\nu_1^{\mathrm{B}}(s,w,q,v)=0$ and we get from Proposition 6.2

$$\sup_{g\in\mathcal{B}_{L_w(Q)}}\left(\mathbb{E}_{\omega_1}\|J_1^{\mathrm{B}}g-A_{\omega_1}^{(1)}(g)\|^w_{B^s_{q^*v^*}(Q)^*}\right)^{1/w}\le cn^{-s/d+1/w-1/q}. \tag{163}$$

Combining (154), (161), (163), and taking into account (160), the result follows.

Case 3: We suppose that (157) holds and

$$\frac{1}{2}<\frac{s}{d}+\frac{1}{q}\le 1.$$

Let w be such that

$$\max\left(\frac{1}{p}-\frac{r}{d},\frac{1}{q},\frac{1}{2}\right)<\frac{1}{w}<\frac{s}{d}+\frac{1}{q}, \tag{164}$$

hence we have $1<w<2$, and thus $\bar{w}=w$. Furthermore,

$$\begin{aligned}\theta(s,w,q) &= \frac{s}{d}-\left(\frac{1}{w}-\frac{1}{q}\right)_+ = \frac{s}{d}+\frac{1}{q}-\frac{1}{w}\\ &\le 1-\frac{1}{w}=1-\frac{1}{\bar{w}} = \tau(w),\end{aligned} \tag{165}$$

where equality between the first and last term holds if and only if

$$\frac{s}{d}+\frac{1}{q}=1. \tag{166}$$

Case 3.1: Assume that (166) does not hold. Then $\theta(s,w,q)<\tau(w)$, and, by (164), $w<\infty$. Therefore Proposition 6.2 implies

$$\sup_{g\in\mathcal{B}_{L_w(Q)}}\left(\mathbb{E}_{\omega_1}\|J_1^{\mathrm{B}}g-A_{\omega_1}^{(1)}(g)\|^w_{B^s_{q^*v^*}(Q)^*}\right)^{1/w}\le cn^{-s/d+1/w-1/q}. \tag{167}$$

The required estimate is a consequence of (154), (161), (167), and (160).

Case 3.2: Now we suppose that (166) holds. Together with (157) this implies $p=1$. Then we get from Proposition 6.2

$$\begin{aligned}&\sup_{g\in\mathcal{B}_{L_w(Q)}}\left(\mathbb{E}_{\omega_1}\|J_1^{\mathrm{B}}g-A_{\omega_1}^{(1)}(g)\|^w_{B^s_{q^*v^*}(Q)^*}\right)^{1/w}\\ &\le cn^{-s/d+1/w-1/q}(\log n)^{\nu_1^{\mathrm{B}}(s,w,q,v)}.\end{aligned} \tag{168}$$

To analyze the exponent of the logarithm, we distinguish between two further subcases.

Case 3.2.1: If $v > 1$ and $q < \infty$, we choose w in such a way that,

$$\max\left(\frac{1}{p} - \frac{r}{d}, \frac{1}{q}, \frac{1}{2}, \frac{1}{v}\right) < \frac{1}{w} < 1 = \frac{s}{d} + \frac{1}{q}\,.$$

It follows that $w < v$, therefore we have

$$\nu_1^{\mathrm{B}}(s, w, q, v) = 1. \tag{169}$$

On the other hand, using (166), (146), and $p = 1$, we get

$$\theta(s, p, q) = \frac{s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+ = 0 = \tau(p).$$

Therefore (147) gives $\nu_1^{\mathrm{B}}(s, p, q, v) = 1$. Since by the assumption of case 3.2.1, (151) does not hold, we have by (152)

$$\nu_2^{\mathrm{B}}(\delta, s, p, q, v) = \nu_1^{\mathrm{B}}(s, p, q, v) = 1. \tag{170}$$

Combining (154), (161), (168), (169), (170), and (160) leads to the desired estimate.

Case 3.2.2: Suppose that $v = 1$ or $q = \infty$. Together with (166) and $p = 1$ this means that (151) holds. Here we choose w in such a way that

$$\max\left(\frac{1}{p} - \frac{r}{d}, \frac{1}{q}, \frac{1}{2}, 1 - \delta\right) < \frac{1}{w} < 1 = \frac{s}{d} + \frac{1}{q}\,.$$

Then

$$\nu_1^{\mathrm{B}}(s, w, q, v) = 2 - \frac{1}{w} < 1 + \delta = \nu_2^{\mathrm{B}}(\delta, s, p, q, v),$$

and the result follows similarly to case 3.2.1.

$\square$

To state the counterpart of Theorem 4.2, let $\widetilde{B}^s_{q^*v^*}(Q)$, respectively $\widetilde{H}^s_{q^*}(Q)$, denote the closure of the set of C^∞ functions with support in Q in the norm of $B^s_{q^*v^*}(Q)$, respectively $H^s_{q^*}(Q)$. Then for $1 \leq q, v \leq \infty$,

$$\widetilde{B}^s_{q^*v^*}(Q)^* = B^{-s}_{qv}(Q), \tag{171}$$

and for $1 < q < \infty$

$$\widetilde{H}^s_{q^*}(Q)^* = H^{-s}_q(Q), \tag{172}$$

with equivalence of norms (see [18], the theorem and relation (12) in section 2.11.2, for the spaces on $\mathbb{R}^d$, and [19, 20] for the passage to bounded Lipschitz domains).

Let

$$U^{\mathrm{B}} : \widetilde{B}^s_{q^*v^*}(Q) \to B^s_{q^*v^*}(Q), \quad U^{\mathrm{H}} : \widetilde{H}^s_{q^*}(Q) \to H^s_{q^*}(Q)$$

be the identical embeddings, and put, analogously to (115),

$$\begin{aligned} \tilde{J}^{\mathrm{B}} &= \left(U^{\mathrm{B}}\right)^* J^{\mathrm{B}} : B^r_{pu}(Q) \to B^{-s}_{qv}(Q), \\ \tilde{J}^{\mathrm{H}} &= \left(U^{\mathrm{H}}\right)^* J^{\mathrm{H}} : H^r_p(Q) \to H^{-s}_q(Q). \end{aligned}$$

Theorem 6.4. *Assume r, s, p, q, u, v satisfy (148), (149), and (150), and let $\delta > 0$. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ with $n \geq 2$*

$$\begin{aligned} c_1 n^{-\gamma} &\leq e^{\mathrm{ran}}_n(\tilde{J}^{\mathrm{B}}, \mathcal{B}_{B^r_{pu}(Q)}, B^{-s}_{qv}(Q)) \\ &\leq e^{\mathrm{ran}}_n(J^{\mathrm{B}}, \mathcal{B}_{B^r_{pu}(Q)}, B^s_{q^*v^*}(Q)^*) \leq c_2 n^{-\gamma}(\log n)^{\nu^{\mathrm{B}}(\delta)}, \end{aligned}$$

and

$$\begin{aligned} c_1 n^{-\gamma} &\leq e^{\mathrm{ran}}_n(\tilde{J}^{\mathrm{H}}, \mathcal{B}_{H^r_p(Q)}, H^{-s}_q(Q)) \\ &\leq e^{\mathrm{ran}}_n(J^{\mathrm{H}}, \mathcal{B}_{H^r_p(Q)}, H^s_{q^*}(Q)^*) \leq c_2 n^{-\gamma}(\log n)^{\nu}, \end{aligned}$$

where

$$\begin{aligned} \gamma &= \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d} + 1 - \frac{1}{\bar{p}}\right), \\ \nu^{\mathrm{B}}(\delta) &= \begin{cases} \nu^{\mathrm{B}}_2(\delta) & \text{if} \quad \gamma > 0, \\ 0 & \text{if} \quad \gamma = 0, \end{cases} \end{aligned}$$

with $\nu^{\mathrm{B}}_2(\delta)$ given by (152) and ν being defined in (116).

Proof. The upper bounds result from Proposition 6.3 and the boundedness of J^{B} and J^{H}. The lower bounds can be derived as in the proof of Theorem 4.2, observing that with a suitable choice of ψ, the analogues of (119) also hold for B^r_{pu} and H^r_p, with $r \in \mathbb{R}$, $r > 0$, see [6], Th. 2.3.2. □

The second statement of Theorem 6.4 together with relation (64) of Theorem 5.2 of [11] solves (up to logarithmic factors) a problem posed by Novak and Woźniakowski, see [15], section 4.3.3, Problem 25, for the case of standard information.

7 Weak solution of elliptic PDE

Here we apply the results obtained above to the randomized complexity of weak solution of elliptic partial differential equations. For such use of approximation results in the deterministic case we refer to [3, 4, 5, 21]. Let Q be a bounded

Lipschitz domain, let $m \in \mathbb{N}$, and consider the bilinear form a on $W_2^m(Q)$ given for $g, h \in W_2^m(Q)$ by

$$a(g,h) = \sum_{|\alpha|,|\beta| \le m} \int_D a_{\alpha\beta}(x) D^\alpha g(x) D^\beta h(x) dx,$$

where $a_{\alpha\beta} \in C(\bar{Q})$. It follows that there is a constant $c_1 > 0$ such that

$$|a(g,h)| \le c_1 \|g\|_{W_2^m(Q)} \|h\|_{W_2^m(Q)} \quad (g, h \in W_2^m(Q)). \tag{173}$$

Furthermore, we assume that there is a constant $c_2 > 0$ such that

$$|a(g,g)| \ge c_2 \|g\|^2_{W_2^m(Q)} \quad (g \in \widetilde{W}_2^m(Q)), \tag{174}$$

that is, a is $\widetilde{W}_2^m(Q)$-elliptic (see, e.g., [22] for notions and background). We consider solving the weak problem associated with the bilinear form a: Given $f \in W_2^{-m}(Q)$, find $z \in \widetilde{W}_2^m(Q)$ such that for all $h \in \widetilde{W}_2^m(Q)$

$$a(z,h) = f(h). \tag{175}$$

It follows from (173) and (174) that the problem has a unique solution $z = S_0 f \in \widetilde{W}_2^m(Q)$ and the solution operator $S_0 : W_2^{-m}(Q) \to \widetilde{W}_2^m(Q)$ is an isomorphism.

Now let $r \in \mathbb{N}_0$ and $1 \le p \le \infty$. We formulate the following condition

$$r = 0,\ p = 1. \tag{176}$$

We assume that

$$\left.\begin{array}{ll} & \text{(176) holds and } \frac{m}{d} > \frac{1}{2} \\ \text{or} & \\ & \text{(176) does not hold, and } \frac{r+m}{d} \ge \left(\frac{1}{p} - \frac{1}{2}\right)_+, \end{array}\right\} \tag{177}$$

hence, by (95), (115), and (127), the embedding $\tilde{J} : W_p^r(Q) \to W_2^{-m}(Q)$ is well-defined and continuous. We consider solving the weak problem (175) for $f \in W_p^r(Q)$. The respective solution operator is $S = S_0 \tilde{J}$, that is

$$S : W_p^r(Q) \xrightarrow{\tilde{J}} W_2^{-m}(Q) \xrightarrow{S_0} \widetilde{W}_2^m(Q).$$

Using the isomorphism property of S_0 and Corollary 4.3 above we immediately get

Corollary 7.1. *Let $r \in \mathbb{N}_0$, $1 \le p \le \infty$ satisfying (177). Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ with $n \ge 2$*

$$c_1 n^{-\gamma} \le e_n^{\rm ran}(S, \mathcal{B}_{W_p^r(Q)}, \widetilde{W}_2^m(Q)) \le c_2 n^{-\gamma} (\log n)^\nu,$$

where

$$\begin{aligned}\gamma &= \min\left(\frac{r+m}{d} - \left(\frac{1}{p} - \frac{1}{2}\right)_+, \frac{r}{d} + 1 - \frac{1}{\bar{p}}\right)\\ \nu &= \begin{cases} \nu_1(m,p,2) & \text{if} \quad \gamma > 0, \\ 0 & \text{if} \quad \gamma = 0, \end{cases}\end{aligned}$$

and ν_1 *is given by (77).*

This complements results on the randomized complexity of elliptic PDE obtained in [9, 10].

In a similar way one can obtain the corresponding results for the deterministic setting, using Theorems 5.1 and 5.2. The respective rates can be read directly from these theorems by setting $q = 2$ and replacing s by m. The case $r/d > 1/p$ is contained in [21], except for the limiting case (105).

Let us compare randomized and deterministic setting just for the case $p = 2$, that is, the right-hand side is supposed to belong to $W_2^r(Q)$ (and the error is measured in $\widetilde{W}_2^m(Q)$). Again we omit logarithmic factors.

$S : W_2^r(Q) \to \widetilde{W}_2^m(Q)$	e_n^{det}	e_n^{ran}
$r/d > 1/2$	$n^{-r/d}$	$n^{-r/d-\min(m/d,1/2)}$
$r/d \leq 1/2$	1	$n^{-r/d-\min(m/d,1/2)}$

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