Randomized Approximation of Sobolev Embeddings II

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Abstract

We study the approximation of Sobolev embeddings by linear randomized algorithms based on function values. Both the source and the target space are Sobolev spaces of non-negative smoothness order, defined on a bounded Lipschitz domain. The optimal order of convergence is determined. We also study the deterministic setting. Using interpolation, we extend the results to other classes of function spaces. In this context a problem posed by Novak and Woźniakowski is solved. Finally, we present an application to the complexity of general elliptic PDE.

1 Introduction

Randomized approximation of functions based on function values was studied by Wasilkowski [22], Novak [10], and Mathé [9]. They considered the approximation of functions from Sobolev spaces $W_p^r(Q)$ in the norm of $L_q(Q)$, under the assumption that $W_p^r(Q)$ is embedded into $C(\bar{Q})$. In this case the rate for randomized approximation is the same as that for the deterministic setting. Recently the case of non-embedding was studied in [7], where it was observed that randomization can bring an essential speedup over deterministic algorithms. In all these papers the target space was $L_q(Q)$ and the domain Q was a cube.

Here we extend the analysis of [7] to the case of Sobolev spaces of non-negative smoothness order as target spaces, and to bounded Lipschitz domains. The paper is a continuation of part I, [7] (target space $L_q(Q)$), and is followed by part III, [8], where the case of a target space with negative smoothness order is studied.

The main results of this paper are proved for Sobolev spaces of integer order. In chapter 5 we use interpolation to extend the results to Besov and Bessel potential spaces. Our methods also give new results for the deterministic setting, extending results of Novak and Triebel [11] and Vybíral [20]. In this connection we solve open problem 18 posed by Novak and Woźniakowski [12].

There is a direct relation to the information complexity of elliptic partial differential equations via regularity and isomorphism theorems. We present the respective consequences.

2 Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{K}$ stand for the field of reals $\mathbb{R}$ or complex numbers $\mathbb{C}$. Let $d \in \mathbb{N}$ and let $Q \subset \mathbb{R}^d$ be a bounded Lipschitz domain. By this we mean for $d = 1$ a finite union of bounded open intervals with disjoint closure. If $d \geq 2$, we mean an open bounded set with a locally Lipschitz boundary. More precisely, for each $x \in \partial Q$ there is an open ball B centered at x, a rotation U of $\mathbb{R}^d$ around x and a Lipschitz function $h : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$Q \cap B = Q \cap U\{(x_1, \dots, x_{d-1}, x_d) \in \mathbb{R}^d \,:\, x_d \leq h(x_1, \dots, x_{d-1})\}$$

(see also [19], Def. 4.3). Throughout the paper we consider $\mathbb{K}$-valued functions and linear spaces over $\mathbb{K}$, with $\mathbb{K}$ being fixed for all the spaces involved. $C(\bar{Q})$ denotes the space of continuous functions on the closure $\bar{Q}$ of Q, endowed with the supremum norm. For $1 \leq p \leq \infty$, let $L_p(Q)$ be the space of $\mathbb{K}$-valued p-integrable functions, equipped with the usual norm

$$\|f\|_{L_p(Q)} = \left(\int_Q |f(x)|^p dx \right)^{1/p}$$

if $p < \infty$, and

$$\|f\|_{L_\infty(Q)} = \operatorname{ess\,sup}_{x \in Q} |f(x)|.$$

Let $r \in \mathbb{N}_0$. The Sobolev space $W_p^r(Q)$ consists of all functions $f \in L_p(Q)$ such that for all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ with $|\alpha| := \sum_{j=1}^d \alpha_j \leq r$, the generalized partial derivative $D^\alpha f$ belongs to $L_p(Q)$. The norm on $W_p^r(Q)$ is defined as

$$\|f\|_{W_p^r(Q)} = \left(\sum_{|\alpha| \leq r} \|D^\alpha f\|_{L_p(Q)}^p \right)^{1/p}$$

if $p < \infty$, and

$$\|f\|_{W_\infty^r(Q)} = \max_{|\alpha| \leq r} \|D^\alpha f\|_{L_\infty(Q)}.$$

For a normed space G the unit ball $\{g \in G \,:\, \|g\| \leq 1\}$ is denoted by $\mathcal{B}_G$. Throughout the paper log means $\log_2$. Furthermore, we often use the same symbol $c, c_1, \dots$ for possibly different positive constants (also when they appear in a sequence of relations). These constants are either absolute or may depend only on the problem parameters p, q, r, s, d and the domain Q, but not on approximation parameters like n, k, l, ω – in all statements of lemmas, propositions, etc., this is precisely described anyway by the order of the quantifiers.

Let $\mathcal{F}(Q)$ denote the linear space of all $\mathbb{K}$-valued functions on Q and let $L_0(Q)$ be the linear space of equivalence classes of Lebesgue measurable functions on Q, with the usual equivalence of being equal except for a set of Lebesgue measure zero. Let $F \subseteq L_0(Q)$ be any nonempty subset and G a normed space. For $n \in \mathbb{N}$ we consider the class $\mathcal{A}_n^{\rm ran}(F, G)$ of linear randomized algorithms from F to G. An element $A \in \mathcal{A}_n^{\rm ran}(F, G)$ has the form

$$A = ((\Omega, \Sigma, \mathbb{P}), (A_\omega)_{\omega\in\Omega}),$$

where $(\Omega, \Sigma, \mathbb{P})$ is a probability space and each A_ω is a linear operator from $\mathcal{F}(Q)$ to G of the form

$$A_\omega(g) = \sum_{i=1}^{n} g(x_{i,\omega})\psi_{i,\omega}$$

with $x_{i,\omega} \in Q$ and $\psi_{i,\omega} \in G$. We assume the following properties: Whenever f_0 and f_1 are representatives of the same class $f \in F \subseteq L_0(Q)$, then

$$A_\omega(f_0) = A_\omega(f_1) \quad \mathbb{P} - \text{a.s.}$$

Furthermore, for each $f \in F$, and each representative f_0 of f the mapping

$$\omega \in \Omega \to A_\omega(f_0)$$

is a random variable with values in G, that is, it is Σ-to-Borel measurable and there is a separable subspace $G_0 \subset G$ (which may depend on f) such that $A_\omega(f_0) \in G_0$ holds $\mathbb{P}$-almost surely. We put $\mathcal{A}^{\rm ran}(F, G) = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n^{\rm ran}(F, G)$.

Let $S : F \to G$ be any mapping. The error of an algorithm $A \in \mathcal{A}_n^{\rm ran}(F, G)$ in approximating S is defined as

$$e(S, A, F, G) = \sup_{f\in F} \mathbb{E} \, \|S(f) - A_\omega(f)\|_G \,,$$

where $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$ (and $+\infty$ is admitted as a possible value). The randomized n-th minimal error (or more precisely, the n-th minimal error with respect to the class of randomized linear algorithms) is defined as

$$e_n^{\rm ran}(S, F, G) = \inf_{A\in\mathcal{A}_n^{\rm ran}(F,G)} e(S, A, F, G).$$

Hence, no linear randomized algorithm that uses at most n function values can provide a smaller error than $e_n^{\rm ran}(S, F, G)$. We have chosen the first moment for the minimal error, which is convenient for the sequel. Statements for other exponents can be read from the proofs below. We also include the target space into the notation since we often consider the same operator acting in different spaces.

We also consider deterministic algorithms. Here we assume that $F \subseteq \mathcal{F}(Q)$ (that is, function values are well-defined). Let G be a normed space. The class of linear deterministic algorithms $\mathcal{A}_n^{\mathrm{det}}(F,G)$ consists of all linear operators from $\mathcal{F}(Q)$ to G of the form

$$A(g) = \sum_{i=1}^{n} g(x_i)\psi_i$$

with $x_i \in Q$ and $\psi_i \in G$. The error of $A \in \mathcal{A}_n^{\mathrm{det}}(F,G)$ in approximating S is defined as

$$e(S,A,F,G) = \sup_{f \in F} \|S(f) - A(f)\|_G$$

and the deterministic n-th minimal error as

$$e_n^{\mathrm{det}}(S,F,G) = \inf_{A \in \mathcal{A}_n^{\mathrm{det}}(F,G)} e(S,A,F,G).$$

The quantities $e_n^{\mathrm{det}}(S,F,G)$ were also called linear sampling numbers [11]. Thus, the $e_n^{\mathrm{ran}}(S,F,G)$ can be viewed as randomized linear sampling numbers.

Throughout this paper we consider only linear algorithms. Concerning more general algorithm classes, see the remark at the end of section 3.

3 Main results

The following is the main result of this paper and extends a result of [7] for the cube to arbitrary bounded Lipschitz domains. Moreover, in [7] the target space was supposed to be L_q, while here we also consider Sobolev spaces W_q^s.

Let $r,s \in \mathbb{N}_0$, $1 \le p,q \le \infty$, and let Q be a bounded Lipschitz domain. Recall from [1], Th. 5.4, that $W_p^r(Q)$ is continuously embedded into $W_q^s(Q)$ if

$$\left.\begin{array}{llll} & 1 \le q < \infty & & \text{and} \quad \frac{r-s}{d} \ge \left(\frac{1}{p} - \frac{1}{q}\right)_+ \\ \text{or} & & & \\ & q = \infty, & 1 < p < \infty, & \text{and} \quad \frac{r-s}{d} > \frac{1}{p} \\ \text{or} & & & \\ & q = \infty, & p \in \{1,\infty\}, & \text{and} \quad \frac{r-s}{d} \ge \frac{1}{p}. \end{array}\right\} \qquad (1)$$

Here we used the notation $a_+ = \max(a,0)$ for $a \in \mathbb{R}$. Let $J : W_p^r(Q) \to W_q^s(Q)$ be the embedding operator.

Theorem 3.1. *Let $r,s \in \mathbb{N}_0$, $1 \le p,q \le \infty$, Q be a bounded Lipschitz domain, and assume that (1) holds. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$*

$$c_1 n^{-\gamma} \le e_n^{\mathrm{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \ \le \ c_2 n^{-\gamma},$$

where

$$\gamma = \frac{r-s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+ .$$

For the proof we need some preparations. Let $0 < \delta < 1$,

$$\varrho \in \mathbb{N}_0, \quad \varrho \ge r - 1, \tag{2}$$

and let

$$Pf = \sum_{j=1}^{\kappa} f(y_j)\psi_j$$

be for $d = 1$ the Lagrange interpolation operator of degree ϱ and for $d > 1$ its tensor product, with $(y_j)_{j=1}^{\kappa}$ the uniform grid on $[0, 1 - \delta]^d$ and $(\psi_j)_{j=1}^{\kappa}$ the respective Lagrange polynomials, considered as functions on $\mathbb{R}^d$. Clearly,

$$Pg = g \quad (g \in \mathcal{P}_\varrho), \tag{3}$$

where $\mathcal{P}_\varrho$ is the space of polynomials on $\mathbb{R}^d$ of degree not exceeding ϱ.

Put $\Omega = [0, \delta]^d$, let Σ be the σ-algebra of Lebesgue measurable subsets of Ω and $\mathbb{P}$ the normalized on $[0, \delta]^d$ Lebesgue measure. For $\omega \in \Omega = [0, \delta]^d$ put

$$y_{j,\omega} = y_j + \omega, \tag{4}$$

$$\psi_{j,\omega}(x) = \psi_j(x - \omega) \quad (x \in \mathbb{R}^d), \tag{5}$$

and define an operator P_ω by setting for any function $f \in \mathcal{F}\left([0, 1]^d\right)$

$$\left(P_\omega f\right)(x) = \sum_{j=1}^{\kappa} f(y_{j,\omega})\psi_{j,\omega}(x) \quad (x \in \mathbb{R}^d). \tag{6}$$

It follows from (3) that

$$P_\omega g = g \quad (g \in \mathcal{P}_\varrho,\, \omega \in \Omega). \tag{7}$$

Moreover, for $1 \le q < \infty$,

$$(\mathbb{E}\, |f(y_{j,\omega})|^q)^{1/q} \le c\|f\|_{L_q([0,1]^d)} \quad (f \in L_q([0, 1]^d),\, 1 \le j \le \kappa). \tag{8}$$

Let Q be a bounded Lipschitz domain. We fix any axis-parallel cube

$$\tilde{Q} = x_0 + [0, b]^d \quad \text{with} \quad Q \subset \tilde{Q}. \tag{9}$$

For $l \in \mathbb{N}_0$ let

$$\tilde{Q} = \bigcup_{i=1}^{2^{dl}} Q_{li}$$

be the partition of $\tilde{Q}$ into 2^{dl} cubes of sidelength $b2^{-l}$ and of disjoint interior. Let x_{li} denote the point in Q_{li} with minimal coordinates. Introduce the following operators E_{li} and R_{li} from $\mathcal{F}(\mathbb{R}^d)$ to $\mathcal{F}(\mathbb{R}^d)$, by setting for $f \in \mathcal{F}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$

$$(E_{li}f)(x) = f(x_{li} + b2^{-l}x) \tag{10}$$

and

$$(R_{li}f)(x) = f(b^{-1}2^l(x - x_{li})). \tag{11}$$

(If these operators are applied to a function f which is defined on a subset of $\mathbb{R}^d$, we assume that f is extended to all of $\mathbb{R}^d$ by zero.)

Define

$$\mathcal{I}_l = \{i \,:\, 1 \le i \le 2^{dl},\ Q_{li} \subseteq Q\}$$

First we establish a simple geometric property. Let $B(x, \tau)$ denote the closed ball of radius τ around $x \in \mathbb{R}^d$ and $B^0(x, \tau)$ its interior.

Lemma 3.2. *There are constants $a > b\sqrt{d}$ and $l_0 \in \mathbb{N}_0$ such that for all $l \ge l_0$*

$$Q \subseteq \bigcup_{i \in \mathcal{I}_l} B(x_{li}, a2^{-l}). \tag{12}$$

Proof. By elementary geometry, the Lipschitz property (in fact, the slightly weaker cone property, see [1], Ch. IV, for the definition) implies the following: There are constants $\tau_0 > 0$, $0 < \gamma_0 < 1$ such that for all $x \in Q$ and all $0 < \tau \le \tau_0$ there is a $y \in Q$ such that

$$B(y, \gamma_0\tau) \subseteq Q \cap B(x, \tau).$$

We define

$$a = (\gamma_0^{-1} + 1)b\sqrt{d} \tag{13}$$

$$l_0 = \max\left(\left\lceil \log \frac{b\sqrt{d}}{\gamma_0\tau_0} \right\rceil, 0\right). \tag{14}$$

Let $l \ge l_0$ and assume the contrary of (12), that is, there is an $x \in Q$ such that

$$|x - x_{li}| > a2^{-l} \quad (i \in \mathcal{I}_l).$$

Then

$$B\left(x, (a - b\sqrt{d})2^{-l}\right) \cap \bigcup_{i \in \mathcal{I}_l} Q_{li} = \emptyset.$$

By (13) and (14),

$$(a - b\sqrt{d})2^{-l} = \gamma_0^{-1}b\sqrt{d}2^{-l} \le \gamma_0^{-1}b\sqrt{d}2^{-l_0} \le \tau_0,$$

so there is a ball of radius $\gamma_0(a - b\sqrt{d})2^{-l}$ contained in

$$Q \setminus \bigcup_{i \in \mathcal{I}_l} Q_{li}.$$

But such a ball contains an axis-parallel cube of sidelength

$$d^{-1/2}\gamma_0(a - b\sqrt{d})2^{-l+1} = b\,2^{-l+1},$$

and hence, a cube $Q_{lj}^{(0)}$ for some j, a contradiction to the definition of $\mathcal{I}_l$.

□

Now we use this lemma to construct a suitable partition of unity on Q. Let

$$\sigma \in \mathbb{N}_0,\ \sigma \ge s, \tag{15}$$

and let $\eta \in C^\sigma(\mathbb{R}^d)$ be such that $\eta \ge 0$, $\eta > 0$ on $B(0,a/b)$, and $\mathrm{supp}\,(\eta) \subseteq B^0(0,2a/b)$, with a from Lemma 3.2 and b from (9), where $C^\sigma(\mathbb{R}^d)$ denotes the space of functions possessing continuous, bounded partial derivatives up to order σ on $\mathbb{R}^d$. Denote

$$B = B^0(0,2a/b)$$

and, for $l \ge l_0$,

$$B_{li} = B^0(x_{li}, a2^{-l+1}).$$

Clearly,

$$\max_{x\in Q} |\{i \in \mathcal{I}_l : x \in B_{li}\}| \le c. \tag{16}$$

It follows from Lemma 3.2 and (16) that there are constants $c_1, c_2, c_3 > 0$ such that for $l \ge l_0$

$$\sum_{j\in\mathcal{I}_l} R_{lj}\eta(x) \ge c_1 \qquad (x \in Q), \tag{17}$$

moreover, for $s_1 \in \mathbb{N}_0$, $0 \le s_1 \le s$,

$$\|R_{li}\eta\|_{C^{s_1}(\mathbb{R}^d)} \le c_2\, 2^{s_1 l} \qquad (i \in \mathcal{I}_l) \tag{18}$$

and

$$\Big\| \sum_{j\in\mathcal{I}_l} R_{lj}\eta \Big\|_{C^{s_1}(\mathbb{R}^d)} \le c_3\, 2^{s_1 l}. \tag{19}$$

Define for $i \in \mathcal{I}_l$ and $l \ge l_0$ a function η_{li} on Q by setting

$$\eta_{li}(x) = \frac{R_{li}\eta(x)}{\sum_{j\in\mathcal{I}_l} R_{lj}\eta(x)} \qquad (x \in Q).$$

Consequently,

$$\eta_{li}(x) = 0 \quad (x \in Q \setminus B_{li}) \tag{20}$$

and

$$\sum_{i\in\mathcal{I}_l} \eta_{li}(x) = 1 \qquad (x \in Q). \tag{21}$$

It follows from the definition of η and from (17–19) that for $0 \le s_1 \le s$

$$\|\eta_{li}\|_{C^{s_1}(Q)} \le c\, 2^{s_1 l}. \tag{22}$$

For $l \ge l_0$ and $\omega \in \Omega$ define $P_{l,\omega} : \mathcal{F}(Q) \to W_q^s(Q)$ by

$$P_{l,\omega} f = \sum_{i\in\mathcal{I}_l} \eta_{li}(R_{li}P_\omega E_{li} f)|_Q \quad (f \in \mathcal{F}(Q)), \tag{23}$$

hence

$$(P_{l,\omega}f)(x) = \sum_{i\in\mathcal{I}_l}\sum_{j=1}^{\kappa} f(x_{li} + b\,2^{-l}y_{j,\omega})\eta_{li}(x)(R_{li}\psi_{j,\omega})(x) \quad (x \in Q). \tag{24}$$

Let

$$A_l = (P_{l,\omega})_{\omega\in\Omega}\,. \tag{25}$$

It easily follows from the definition that

$$A_l \in \mathcal{A}^{\rm ran}_{\kappa 2^{dl}}(W^r_p(Q), W^s_q(Q)). \tag{26}$$

Proposition 3.3. *Let $d \in \mathbb{N}$, $r, s \in \mathbb{N}_0$, $1 \le p, q \le \infty$, let Q be a bounded Lipschitz domain, and assume that (1) is satisfied. Let $(P_{l,\omega})_{\omega\in\Omega}$ for $l \ge l_0$ be given by (23), with parameters ϱ and σ satisfying (2) and (15). Then there is a constant $c > 0$ such that for all $l \ge l_0$ and $f \in W^r_p(Q)$ the following hold.*

If $q < \infty$, then

$$(\mathbb{E}\,\|f - P_{l,\omega}f\|^q_{W^s_q(Q)})^{1/q} \;\le\; c\,2^{-(r-s)l+\max(1/p-1/q,0)dl}\|f\|_{W^r_p(Q)}, \tag{27}$$

and if $q = \infty$, then

$$\operatorname{ess\,sup}_{\omega\in\Omega}\|f - P_{l,\omega}f\|_{W^s_\infty(Q)} \;\le\; c\,2^{-(r-s)l+dl/p}\|f\|_{W^r_p(Q)}. \tag{28}$$

Proof. By (1), we have

$$\|f\|_{W^s_q(B)} \le c\|f\|_{W^r_p(B)} \quad (f \in W^r_p(B)). \tag{29}$$

We show (27), relation (28) follows in the same way, with the usual modifications. It follows from (6), (8) and (29) that for $f \in W^r_p(B)$ and $0 \le s_1 \le s$

$$\begin{aligned}\left(\mathbb{E}\,\|P_\omega f\|^q_{W^{s_1}_q(B)}\right)^{1/q} &\le \left(\mathbb{E}\left(\sum_{j=1}^{\kappa}|f(y_{j,\omega})|\|\psi_{j,\omega}\|_{W^{s_1}_q(B)}\right)^q\right)^{1/q} \\ &\le c\sum_{j=1}^{\kappa}(\mathbb{E}\,|f(y_{j,\omega})|^q)^{1/q} \le c\|f\|_{L_q(B)} \\ &\le c\|f\|_{W^r_p(B)}.\end{aligned} \tag{30}$$

We denote

$$|f|_{r,p,B} = \left(\sum_{|\alpha|=r}\|D^\alpha f\|^p_{L_p(B)}\right)^{1/p}$$

if $p < \infty$ and

$$|f|_{r,\infty,B} = \max_{|\alpha|=r}\|D^\alpha f\|_{L_\infty(B)}.$$

Next we apply Theorem 3.1.1 from [2]: there is a constant $c > 0$ such that for all $f \in W_p^r(B)$

$$\inf_{g \in \mathcal{P}_\varrho} \|f - g\|_{W_p^r(B)} \le c|f|_{r,p,B}. \tag{31}$$

It follows from (7), and (29–31) that

$$\begin{aligned} (\mathbb{E}\,\|f - P_\omega f\|^q_{W_q^s(B)})^{1/q} &= \inf_{g \in \mathcal{P}_\varrho} \left(\mathbb{E}\,\|(f-g) - P_\omega(f-g)\|^q_{W_q^{s_1}(B)}\right)^{1/q} \\ &\le c \inf_{g \in \mathcal{P}_\varrho} \|f - g\|_{W_p^r(B)} \le c|f|_{r,p,B}. \end{aligned} \tag{32}$$

Let $f \in W_p^r(Q)$ and let $\tilde{f} \in W_p^r(\mathbb{R}^d)$ be an extension of f with

$$\|\tilde{f}\|_{W_p^r(\mathbb{R}^d)} \le c\|f\|_{W_p^r(Q)}$$

(see [13]). Observe that for $0 \le s_1 \le s$

$$\|R_{li} g\|_{W_q^{s_1}(B_{li})} \le c 2^{(s_1 - d/q)l} \|g\|_{W_q^{s_1}(B)} \quad (g \in W_q^{s_1}(B)) \tag{33}$$

and by (22),

$$\|\eta_{li} g\|_{W_q^s(Q \cap B_{li})} \le c \sum_{s_1=0}^{s} 2^{(s-s_1)l} \|g\|_{W_q^{s_1}(Q \cap B_{li})} \quad (g \in W_q^s(Q \cap B_{li})). \tag{34}$$

Because of (16), (20), and (21) we get

$$\begin{aligned} (\mathbb{E}\,\|f - P_{l,\omega} f\|^q_{W_q^s(Q)})^{1/q} &= \left(\mathbb{E}\left\|\sum_{i \in \mathcal{I}_l} \eta_{li}(f - R_{li} P_\omega E_{li} f)\right\|^q_{W_q^s(Q)}\right)^{1/q} \\ &\le c \left(\sum_{i \in \mathcal{I}_l} \mathbb{E}\,\|\eta_{li}(f - R_{li} P_\omega E_{li} f)\|^q_{W_q^s(Q)}\right)^{1/q}. \end{aligned} \tag{35}$$

Furthermore, using (34), (33), and (32),

$$\begin{aligned} \mathbb{E}\,\|\eta_{li}(f - R_{li} P_\omega E_{li} f)\|^q_{W_q^s(Q)} &= \mathbb{E}\,\|\eta_{li}(f - R_{li} P_\omega E_{li} f)\|^q_{W_q^s(Q \cap B_{li})} \\ &\le c \sum_{s_1=0}^{s} 2^{q(s-s_1)l} \mathbb{E}\,\|f - R_{li} P_\omega E_{li} f)\|^q_{W_q^{s_1}(Q \cap B_{li})} \\ &\le c \sum_{s_1=0}^{s} 2^{q(s-s_1)l} \mathbb{E}\,\|\tilde{f} - R_{li} P_\omega E_{li} \tilde{f}\|^q_{W_q^{s_1}(B_{li})} \\ &\le c \sum_{s_1=0}^{s} 2^{(qs-d)l} \mathbb{E}\,\|E_{li}\tilde{f} - P_\omega E_{li} \tilde{f}\|^q_{W_q^{s_1}(B)} \\ &\le c\, 2^{(qs-d)l} |E_{li}\tilde{f}|^q_{r,p,B}. \end{aligned} \tag{36}$$

By Hölder's inequality,

$$
\begin{aligned}
&\left(2^{-dl}\sum_{i\in\mathcal{I}_l}|E_{li}\tilde f|^q_{r,p,B}\right)^{1/q}\\
\le\;& c\,2^{\max(1/p-1/q,0)dl}\left(2^{-dl}\sum_{i\in\mathcal{I}_l}|E_{li}\tilde f|^p_{r,p,B}\right)^{1/p}\\
=\;& c\,2^{\max(1/p-1/q,0)dl}\left(2^{-dl}\sum_{i\in\mathcal{I}_l}\sum_{|\alpha|=r}\int_B\left|D^\alpha\left(\tilde f(x_{li}+b\,2^{-l}x)\right)\right|^p\,dx\right)^{1/p}\\
\le\;& c\,2^{-rl+\max(1/p-1/q,0)dl}\left(\sum_{i\in\mathcal{I}_l}\sum_{|\alpha|=r}\int_{B_{li}}\left|\left(D^\alpha\tilde f\right)(y)\right|^p\,dy\right)^{1/p}\\
=\;& c\,2^{-rl+\max(1/p-1/q,0)dl}\left(\sum_{i\in\mathcal{I}_l}|\tilde f|^p_{r,p,B_{li}}\right)^{1/p}\\
\le\;& c\,2^{-rl+\max(1/p-1/q,0)dl}|\tilde f|_{r,p,\mathbb{R}^d}\le c\,2^{-rl+\max(1/p-1/q,0)dl}\|\tilde f\|_{W_p^r(\mathbb{R}^d)}\\
\le\;& c\,2^{-rl+\max(1/p-1/q,0)dl}\|f\|_{W_p^r(Q)} \qquad (37)
\end{aligned}
$$

(with the usual modifications for $p=\infty$). Combining (35–37) gives

$$(\mathbb{E}\,\|f-P_{l,\omega}f\|^q_{W_q^s(Q)})^{1/q}\le c\,2^{(s-r)l+\max(1/p-1/q,0)dl}\|f\|_{W_p^r(Q)},$$

which concludes the proof of (27). □

Proof of Theorem 3.1. The upper bound is an immediate consequence of Proposition 3.3, (26), and the monotonicity of the n-th minimal error with repect to n.

Now we show the lower bound. Let $Q'=x_0'+[0,b']^d$ be a closed axis-parallel cube contained in Q. Let $\psi\not\equiv 0$ be a C^∞ function on $\mathbb{R}^d$ with support in the interior of $[0,1]^d$. Let $n\in\mathbb{N}$, and put

$$k=\left\lceil\frac{\log n+1}{d}\right\rceil,$$

hence

$$2^{d(k-1)}<2n\le 2^{dk}.$$

Put

$$\psi_i=R'_{ki}\psi\quad(1\le i\le 2^{dk}),$$

where R'_{ki} is defined by analogy to (11), with $\tilde Q$ replaced by Q'. Observe that

$$c_1\,2^{rk-dk/p}\,\|(\alpha_i)\|_{\ell_p^{2^{dk}}}\le\left\|\sum_{i=1}^{2^{dk}}\alpha_i\psi_i\right\|_{W_p^r(Q)}\le c_2\,2^{rk-dk/p}\,\|(\alpha_i)\|_{\ell_p^{2^{dk}}} \qquad (38)$$

for all $(\alpha_i) \in \mathbb{K}^{2^{dk}}$, where

$$\|(\alpha_i)\|_{\ell_p^{2^{dk}}} = \left(\sum_{i=1}^{2^{dk}} |\alpha_i|^p \right)^{1/p} .$$

Using the well-known relation between randomized and average minimal error (see [10, 14, 4]), here with respect to the counting measure on

$$\left\{ \pm \|\psi_i\|_{W_p^r(Q)}^{-1} \psi_i \; : \; i = 1, \ldots, 2^{dk} \right\},$$

we obtain, taking into account relation (38),

$$\begin{aligned} e_n^{\mathrm{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) &\geq \frac{2^{dk} - n}{2^{dk}} \min_{1 \leq i \leq 2^{dk}} \frac{\|J\psi_i\|_{W_q^s(Q)}}{\|\psi_i\|_{W_p^r(Q)}} \\ &\geq c\, 2^{sk - dk/q - rk + dk/p} = c\, 2^{-(r-s)k + (1/p - 1/q)dk} \\ &\geq c n^{-(r-s)/d + 1/p - 1/q}. \end{aligned} \tag{39}$$

Now we prove a second estimate. Let ε_i $(i = 1, \ldots, 2^{dk})$ be independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = 1/2$. We use again the average minimal error, this time with respect to the distribution of

$$M_k^{-1} \sum_{i=1}^{2^{dk}} \varepsilon_i \psi_i,$$

where

$$M_k = \max \left\{ \left\| \sum_{i=1}^{2^{dk}} \alpha_i \psi_i \right\|_{W_p^r(Q)} \; : \; \alpha_i \in \{-1, 1\}, i = 1, \ldots, 2^{dk} \right\}.$$

Combined with (38), we get

$$\begin{aligned} & e_n^{\mathrm{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \\ & \geq M_k^{-1} \min \left\{ \mathbb{E} \left\| \sum_{i \in \mathcal{I}} \varepsilon_i J \psi_i \right\|_{W_q^s(Q)} \; : \; \mathcal{I} \subseteq \{1, \ldots, 2^{dk}\}, \; |\mathcal{I}| \geq 2^{dk} - n \right\} \\ & \geq c\, 2^{-rk + sk - dk/q} (2^{dk} - n)^{1/q} \geq c n^{-(r-s)/d}. \end{aligned} \tag{40}$$

Now the lower bound in Theorem 3.1 is a consequence of (39) and (40).

□

Remark. By the same technique it can be shown that the lower bounds in Theorem 3.1 also hold for the n-th minimal errors defined with respect to the class of randomized adaptive nonlinear algorithms (see e.g. [5, 6] for these notions). On the other hand, upper bounds for a given algorithm class automatically hold for any larger algorithm class. It follows that the rate in Theorem 3.1 also holds for the class of randomized adaptive nonlinear algorithms, and also for any class in between. In particular, it holds for randomized (nonlinear) sampling numbers [7], as well.

4 Deterministic setting

Next we show the analogue of Proposition 3.3 for the deterministic case. First we consider the case that $W_p^r(Q)$ is continuously embedded into $C(\bar{Q})$. For $r \in \mathbb{N}_0$ this holds if and only if

$$\left.\begin{array}{lll} & p=1 & \text{and} \quad r/d \ge 1 \\ \text{or} & & \\ & 1 < p \le \infty & \text{and} \quad r/d > 1/p, \end{array}\right\} \tag{41}$$

see [1], Ch. 5. In these cases we consider $W_p^r(Q)$ as identified with a subset of $C(\bar{Q})$, hence, function values at points of Q are well-defined and deterministic algorithms as introduced in section 2 make sense. In particular, setting $\omega = 0$ in (23), we obtain a deterministic linear algorithm $P_{l,0}$.

The following is the deterministic counterpart of Proposition 3.3.

Proposition 4.1. *Let $d \in \mathbb{N}$, $r, s \in \mathbb{N}_0$, $1 \le p, q \le \infty$, let Q be a bounded Lipschitz domain, and assume that (1) and (41) are satisfied. Let $P_{l,0}$ for $l \ge l_0$ be given by (23) with $\omega = 0$ and parameters ϱ and σ satisfying (2) and (15). Then there is a constant $c > 0$ such that for all $l \ge l_0$ and $f \in W_p^r(Q)$ the following holds:*

$$\sup_{f \in \mathcal{B}_{W_p^r(Q)}} \|f - P_{l,0} f\|_{W_q^s(Q)} \le c\, 2^{-(r-s)l + \max(1/p - 1/q, 0) dl}. \tag{42}$$

Proof. As in the proof of Proposition 3.3 we put $B = B^0(0, 2a/b)$, with a from Lemma 3.2 and b from (9). Since $W_p^r(B)$ is continuously embedded into $C(\bar{B})$, we have the following instead of (30). For $f \in W_p^r(B)$ and $0 \le s_1 \le s$

$$\begin{aligned} \|P_0 f\|_{W_q^{s_1}(B)} &\le \sum_{j=1}^{\kappa} |f(y_j)| \|\psi_j\|_{W_q^{s_1}(B)} \\ &\le c \sum_{j=1}^{\kappa} |f(y_j)| \le c\|f\|_{C(\bar{B})} \le c\|f\|_{W_p^r(B)}. \end{aligned}$$

From (16), (20), and (21) we get

$$\begin{aligned} \|f - P_{l,0}f\|_{W_q^s(Q)} &= \Big\| \sum_{i\in\mathcal{I}_l} \eta_{li}(f - R_{li}P_0E_{li}f) \Big\|_{W_q^s(Q)} \\ &\le c \left(\sum_{i\in\mathcal{I}_l} \|\eta_{li}(f - R_{li}P_0E_{li}f)\|^q_{W_q^s(Q)} \right)^{1/q} . \end{aligned}$$

The rest of the proof of (42) is essentially the same as that of (27). □

Theorem 4.2. *Let $r, s \in \mathbb{N}_0$, $1 \le p, q \le \infty$, Q be a bounded Lipschitz domain, and assume that (1) and (41) hold. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$*

$$c_1 n^{-\gamma} \le e_n^{\det}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \le c_2 n^{-\gamma},$$

where

$$\gamma = \frac{r-s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+ .$$

Proof. The upper bound follows from Proposition 4.1. The lower bound can be obtained by standard techniques for the deterministic setting (see [10], [14]), based on relation (38). We omit details. □

Comparing randomized and deterministic setting for the case of the embedding condition (41), we see that randomization gives no speedup.

Now consider the case that (41) does not hold, hence $W_p^r(Q)$ is not embedded into $C(\bar{Q})$. In this case values of $W_p^r(Q)$ functions are not well-defined, so $e_n^{\det}$ makes no sense. This changes, if instead of $\mathcal{B}_{W_p^r(Q)}$ we consider the dense subset $\mathcal{B}_{W_p^r(Q)} \cap C(\bar{Q})$. Then function values are defined. However, deterministic algorithms do not give any non-trivial convergence rate at all, as the following result shows. It extends Proposition 2 of [7], where the case $s = 0$ was considered.

Theorem 4.3. *Let $r, s \in \mathbb{N}_0$, $1 \le p, q \le \infty$ and assume that (1) holds, but (41) does not. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$*

$$c_1 \le e_n^{\det}(J, \mathcal{B}_{W_p^r(Q)} \cap C(\bar{Q}), W_q^s(Q)) \le c_2. \tag{43}$$

Proof. The upper bound is just the boundedness of J. To show the lower bound, note that (41) does not hold iff

$$p = 1 \quad \text{and} \quad r/d < 1 \tag{44}$$

or

$$1 < p < \infty \quad \text{and} \quad r/d \le 1/p \tag{45}$$

or

$$p = \infty \quad \text{and} \quad r = 0. \tag{46}$$

If (46) holds, then (1) implies $s = 0$, so J is the embedding of $L_\infty(Q)$ into $L_q(Q)$ $(1 \le q \le \infty)$. Consequently,

$$e_n^{\det}(J, \mathcal{B}_{L_\infty(Q)} \cap C(\bar{Q}), L_q(Q)) \quad = \quad e_n^{\det}(J, \mathcal{B}_{C(\bar{Q})}, L_q(Q)) \ge c > 0,$$

since the embedding of $C(\bar{Q})$ into $L_q(Q)$ is not compact. If (44) or (45) hold, then the conditions of Lemma 1 in [7] are satisfied and the same argument as in the proof of Proposition 2 of [7] gives the lower bound of (43). $\square$

Comparing deterministic and randomized setting we conclude that in this case randomization can give a speedup of up to $n^{-\beta}$ for any β with $0 < \beta < 1$.

5 Extension to other function spaces

So far we considered Sobolev spaces whose smoothness order is a nonnegative integer. Now we show how to extend the results to other function spaces by interpolation. For $r \in \mathbb{R}, r \ge 0$, $1 \le p, u \le \infty$, let $B^r_{pu}(Q)$ denote the Besov space and for $1 < p < \infty$ let $H^r_p(Q)$ be the Bessel potential space (also called fractional Sobolev space). For the definition of these spaces on $\mathbb{R}^d$ we refer to [15, 16] and for the case of bounded Lipschitz domains to [17, 19]. Throughout this section we consider only complex-valued functions and spaces over the complex numbers (see, however, the remark at the end of this section for the real case).

We use the following relations between these function spaces.

$$H^r_p(Q) = W^r_p(Q) \quad (r \in \mathbb{N}_0,\ 1 < p < \infty), \tag{47}$$

where equality is meant as algebraic identity with equivalence of norms, see [19], (1.9),

$$B^r_{p,1}(Q) \subset W^r_p(Q) \subset B^r_{p,\infty}(Q) \quad (p = 1, \infty,\ r \in \mathbb{N}_0), \tag{48}$$

the notation $\subset$ meaning algebraic inclusion with continuous embedding, see [19], (4.22)–(4.25),

$$B^0_{\infty,1}(Q) \subset C(\bar{Q}) \subset B^0_{\infty,\infty}(Q), \tag{49}$$

see [19], (4.25), and for $r \in \mathbb{R}$, $r \ge 0$, $1 < p < \infty$,

$$B^r_{p,\min(p,2)}(Q) \subset H^r_p(Q) \subset B^r_{p,\max(p,2)}(Q), \tag{50}$$

see [19], (1.8), (1.299). We also use the following interpolation results. Let

$$r_0, r_1 \in \mathbb{R},\ r_0, r_1 \ge 0,\ 0 < \theta < 1,\ r = (1-\theta)r_0 + \theta r_1.$$

For $1 \le p, u_0, u_1, u \le \infty$, $r_0 \ne r_1$,

$$(B^{r_0}_{p,u_0}(Q), B^{r_1}_{p,u_1}(Q))_{\theta,u} = B^r_{pu}(Q), \tag{51}$$

where $(\, , \,)_{\theta,u}$ denotes real interpolation, see [19], Cor. 1.111, relation (1.368), and for $1 < p_0, p_1 < \infty$, with p given by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

we have

$$\left[H^{r_0}_{p_0}(Q), H^{r_1}_{p_1}(Q)\right]_\theta = H^r_p(Q), \tag{52}$$

where $[\, , \,]_\theta$ denotes classical complex interpolation, see [19], Cor. 1.111, relation (1.372).

Let $(\Omega, \Sigma, \mathbb{P})$ be the probability space from section 2, defined before relation (4). For $1 \le p < \infty$ and a Banach space X we denote by $L_p(\Omega, X)$ the space of X-valued p-th power Bochner integrable functions on $(\Omega, \Sigma, \mathbb{P})$. For $r, s \in \mathbb{N}_0$, $1 \le p, q \le \infty$ satisfying (1), $\varrho \ge r-1$, $\sigma \ge s$ (ϱ, σ the parameters from (2) and (15)), $1 \le q_1 < \infty$, $q_1 \le q$, $l \ge l_0$, define the operators

$$P_l, \, I : W^r_p(Q) \to L_{q_1}(\Omega, W^s_q(Q))$$

by setting for $\omega \in \Omega$

$$\begin{aligned} (P_l f)(\omega) &= P_{l,\omega} f \\ (If)(\omega) &= f. \end{aligned}$$

By Proposition 3.3 and (1), these operators are well-defined, bounded, and we have

$$\|I - P_l : W^r_p(Q) \to L_{q_1}(\Omega, W^s_q(Q))\| \le c\, 2^{-(r-s)l + \max(1/p-1/q,0)dl}. \tag{53}$$

We start with the counterpart of Proposition 3.3.

Proposition 5.1. *Let $r, s \in \mathbb{R}$, $r > s \ge 0$, $1 \le p, q \le \infty$, $1 \le q_1 < \infty$, $q_1 \le q$, $1 \le u, v \le \infty$, and assume $(r-s)/d > 1/p - 1/q$. Let Q be a bounded Lipschitz domain. Let $(P_{l,\omega})_{\omega\in\Omega}$ for $l \ge l_0$ be given by (23), where we assume that the involved parameters ϱ and σ from (2) and (15) satisfy $\varrho \ge r$, $\sigma \ge r+1$. Then there is a constant $c > 0$ such that for all $l \ge l_0$ the following hold. If $s > 0$*

$$\sup_{f \in \mathcal{B}_{B^r_{pu}(Q)}} \left(\mathbb{E}\, \|f - P_{l,\omega} f\|^{q_1}_{B^s_{qv}(Q)}\right)^{1/q_1} \le c\, 2^{-(r-s)l + \max(1/p-1/q,0)dl}, \tag{54}$$

furthermore, for $s = 0$,

$$\sup_{f \in \mathcal{B}_{B^r_{pu}(Q)}} \left(\mathbb{E}\, \|f - P_{l,\omega} f\|^{q_1}_{L_q(Q)}\right)^{1/q_1} \le c\, 2^{-rl + \max(1/p-1/q,0)dl}, \tag{55}$$

and finally, if $s \ge 0$ and $1 < p, q < \infty$,

$$\sup_{f \in \mathcal{B}_{H^r_p(Q)}} \left(\mathbb{E}\, \|f - P_{l,\omega} f\|^{q_1}_{H^s_q(Q)}\right)^{1/q_1} \le c\, 2^{-(r-s)l + \max(1/p-1/q,0)dl}. \tag{56}$$

Proof. We prove the case of Besov spaces by the help of real interpolation. The case of Bessel potential spaces can be handled in the same way using complex interpolation. We start with the case $p = q$. Put $r_0 = \lceil r \rceil - 1$, $r_1 = \lfloor r \rfloor + 1$, hence $0 \le r_0 < r < r_1$. Moreover, $\varrho \ge r_1 - 1$ and $\sigma \ge r_1$. Let $0 < \theta < 1$ be such that $r = (1-\theta)r_0 + \theta r_1$. By (47–48), (50–51), we have

$$(W_p^{r_0}(Q), W_p^{r_1}(Q))_{\theta,u} = B_{pu}^r(Q). \tag{57}$$

Furthermore, if X_0, X_1 is an interpolation pair of Banach spaces, then

$$(L_{q_1}(\Omega, X_0), L_{q_1}(\Omega, X_1))_{\theta,u} = L_{q_1}(\Omega, (X_0, X_1)_{\theta,u}), \tag{58}$$

and the respective statement holds for complex interpolation (see [15], 1.18.4). By (53),

$$\|I - P_l : W_p^{r_i}(Q) \to L_{q_1}(\Omega, L_p(Q))\| \le c\, 2^{-r_i l} \quad (i = 0, 1).$$

Using (57), interpolation gives

$$\|I - P_l : B_{pu}^r(Q) \to L_{q_1}(\Omega, L_p(Q))\| \le c\, 2^{-rl}, \tag{59}$$

which proves relation (55) for $p = q$. Moreover, again from (53),

$$\|I - P_l : W_p^{r_i}(Q) \to L_{q_1}(\Omega, W_p^{r_i}(Q))\| \le c \quad (i = 0, 1),$$

which by (57) and (58) implies

$$\|I - P_l : B_{pu}^r(Q) \to L_{q_1}(\Omega, B_{pu}^r(Q))\| \le c. \tag{60}$$

Next define $\theta = s/r$ and let $1 \le v \le \infty$. By (47–48), (50–51), we have

$$(L_p(Q), B_{pu}^r(Q))_{\theta,v} = B_{pv}^s(Q).$$

The interpolation property together with (58), (59), and (60) gives

$$\|I - P_l : B_{pu}^r(Q) \to L_{q_1}(\Omega, B_{pv}^s(Q))\| \le c\, 2^{-(r-s)l}. \tag{61}$$

This implies (54) for $p = q$.

Now assume $1 \le p \ne q \le \infty$, $r > s \ge 0$, $(r-s)/d > 1/p - 1/q$. Let

$$r_1 = r - d\left(\frac{1}{p} - \frac{1}{q}\right)_+ .$$

It follows that $r_1 > s$. We use that the following embedding is continuous:

$$B_{pu}^r(Q) \to B_{qu}^{r_1}(Q).$$

(For $p < q$ this follows from [16], Theorem 2.7.1, and the remarks in Ch. 2.3 of [17] about passing from $\mathbb{R}^d$ to arbitrary domains. For $p \ge q$ it is a consequence of [19], Th. 1.118.) With this, the first two statements of Proposition 5.1 for the case $p \neq q$ can be derived from those for $p = q$, which we show for (54), relation (55) follows analogously. We conclude from (61) (with r_1 in place of r and q in place of p) that

$$\begin{aligned} &\|I - P_l : B^r_{pu}(Q) \to L_{q_1}(\Omega, B^s_{qv}(Q))\| \\ \leq \quad & \|I : B^r_{pu}(Q) \to B^{r_1}_{qu}(Q)\| \|I - P_l : B^{r_1}_{qu}(Q) \to L_{q_1}(\Omega, B^s_{qv}(Q))\| \\ \leq \quad & c\, 2^{-(r_1 - s)l} = c\, 2^{-(r-s)l + \max(1/p - 1/q, 0)dl}. \end{aligned}$$

□

Theorem 5.2. *Assume $r, s \in \mathbb{R}$, $r > s \geq 0$, $1 \leq p, q \leq \infty$, $1 \leq u, v \leq \infty$, and $(r-s)/d > 1/p - 1/q$. Then there are constants $c_1, c_2 > 0$ such that the following holds for all $n \in \mathbb{N}$. If $s > 0$, then*

$$c_1 n^{-\gamma} \leq e^{\mathrm{ran}}_n(J, \mathcal{B}_{B^r_{pu}(Q)}, B^s_{qv}(Q)) \leq c_2 n^{-\gamma}, \tag{62}$$

if $s = 0$, then

$$c_1 n^{-\gamma} \leq e^{\mathrm{ran}}_n(J, \mathcal{B}_{B^r_{pu}(Q)}, L_q(Q)) \leq c_2 n^{-\gamma}, \tag{63}$$

and if $s \geq 0$ and $1 < p, q < \infty$, then

$$c_1 n^{-\gamma} \leq e^{\mathrm{ran}}_n(J, \mathcal{B}_{H^r_p(Q)}, H^s_q(Q)) \leq c_2 n^{-\gamma}, \tag{64}$$

where J stands for the respective embedding operator, and

$$\gamma = \frac{r-s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+ .$$

Proof. The upper bounds follow from Proposition 5.1. To obtain the lower bounds, note that with a suitable choice of ψ, the analogues of (38) also hold for B^r_{pu} and H^r_p instead of W^r_p, see [3], Th. 2.3.2. Therefore, the lower bound proof of Theorem 3.1 goes through with the proper changes. □

Relation (64) gives a partial solution to Problem 25 of Novak and Woźniakowski [12], section 4.3.3. It settles the case of standard information with $s \geq 0$. The case of standard information with $s < 0$ is studied in [8].

A similar remark as that made at the end of section 3 applies to Theorem 5.2, as well.

We can also extend Theorem 4.2 to other function spaces via interpolation. Here, however, a somewhat more involved approach than in the randomized setting is required, since we have to ensure the condition of embedding into $C(\bar{Q})$ also for the spaces to be interpolated.

The case $s = 0$ of Theorem 5.3 below is due to Novak and Triebel [11]. Results for the case $s > 0$ are given for the cube by Vybíral [20]. For bounded

Lipschitz domains these rates were established for the nonlinear sampling numbers by Triebel [18]. Our result shows that they also hold for the case of linear sampling numbers, this way solving Problem 18 of Novak and Woźniakowski [12], section 4.2.4.

The case of Besov spaces $B^s_{qv}(Q)$ with $s = 0$, which is not covered by Theorem 5.3, is studied for the cube in [21].

Theorem 5.3. *Under the assumptions of Theorem 5.2 and the additional assumption $r/d > 1/p$ the results (62-64) stated there also hold with $e_n^{\rm det}$ in place of $e_n^{\rm ran}$.*

The derivation of the lower bounds is again quite standard and uses the facts indicated in the proof of Theorem 5.2. We omit it here. The upper bounds are a consequence of the following analogue of Proposition 4.1.

Proposition 5.4. *Let $r, s \in \mathbb{R}$, $r > s \ge 0$, $1 \le p, q \le \infty$, $1 \le u, v \le \infty$, and assume $r/d > 1/p$ and $(r-s)/d > 1/p - 1/q$. Let Q be a bounded Lipschitz domain. Let $P_{l,0}$ for $l \ge l_0$ be given by (23) with $\omega = 0$ and the parameters $\varrho, \sigma \in \mathbb{N}_0$ from (2) and (15) satisfying*

$$\varrho > r, \quad \sigma > r + 1. \tag{65}$$

Then there is a constant $c > 0$ such that for all $l \ge l_0$ the following hold. For $s > 0$

$$\sup_{f \in \mathcal{B}_{B^r_{pu}(Q)}} \|f - P_{l,0} f\|_{B^s_{qv}(Q)} \le c\, 2^{-(r-s)l + \max(1/p - 1/q, 0)dl}, \tag{66}$$

for $s = 0$

$$\sup_{f \in \mathcal{B}_{B^r_{pu}(Q)}} \|f - P_{l,0} f\|_{L_q(Q)} \le c\, 2^{-rl + \max(1/p - 1/q, 0)dl}, \tag{67}$$

and for $s \ge 0, 1 < p, q < \infty$

$$\sup_{f \in \mathcal{B}_{H^r_p(Q)}} \|f - P_{l,0} f\|_{H^s_q(Q)} \le c\, 2^{-(r-s)l + \max(1/p - 1/q, 0)dl}. \tag{68}$$

Proof. The case $s = 0$ follows from results of Novak and Triebel [11] (see also [19], section 4.3.3). They prove their Proposition 22, or more precisely, relations (4.30) and (4.35), under certain general assumptions on the approximating operators, see [11], relations (4.22) – (4.25) and (2.67). That these conditions are also satisfied for $P_{l,0}$ given by (24) follows from (7), (16), (20), (21), (23), and (65) (condition (4.24) of [11] is satisfied with a general constant $c > 0$ instead of constant 2, which, however, does not affect the result).

Now we deal with the case $s > 0$. Similarly to the randomized setting we discuss only the case $p = q$, the general case follows by the same embedding argument as in the proof of Proposition 5.1.

First we assume $r > d$. Put $r_0 = \lceil r \rceil - 1$, $r_1 = \lfloor r \rfloor + 1$. Then $d \le r_0 < r < r_1$, and the embedding condition (41) is satisfied for r_0 and all $1 \le p \le \infty$. We have from (42)

$$\|I - P_{l,0} : W_p^{r_i}(Q) \to W_p^{r_i}(Q)\| \le c \quad (i = 0, 1). \tag{69}$$

Choose θ in such a way that $r = (1-\theta)r_0 + \theta r_1$. Using (47–48), (50–51), we interpolate (69) and get

$$\|I - P_{l,0} : B^r_{pu}(Q) \to B^r_{pu}(Q)\| \le c. \tag{70}$$

By (67),

$$\|I - P_{l,0} : B^r_{pu}(Q) \to L_p(Q)\| \le c\, 2^{-rl}. \tag{71}$$

Setting $\theta = s/r$ and interpolating (70) and (71), we obtain (66) for $r > d$. The case of Bessel potential spaces (68) can be derived in an analogous way, using complex interpolation and the (already proven) case $s = 0$ of (68).

Next we consider the case $r \le d$, $p = \infty$ (thus, we have to deal with the case of Besov spaces only). Here we note that

$$\|I - P_{l,0} : C(\bar{Q}) \to L_\infty(Q)\| \le c, \tag{72}$$

which easily follows from the definition (24) of $P_{l,0}$ and from (16) and (20). Put $r_1 = \lfloor r \rfloor + 1$. By (42),

$$\|I - P_{l,0} : W^{r_1}_\infty(Q) \to W^{r_1}_\infty(Q)\| \le c. \tag{73}$$

Taking into account (48) and (49), interpolation of (72) and (73) gives

$$\|I - P_{l,0} : B^r_{\infty,u}(Q) \to B^r_{\infty,u}(Q)\| \le c. \tag{74}$$

Using again (67), we have

$$\|I - P_{l,0} : B^r_{\infty,u}(Q) \to L_\infty(Q)\| \le c\, 2^{-rl}, \tag{75}$$

and interpolating (74) and (75) with $\theta = s/r$ we get (66).

Finally, let $r \le d$ and $1 < p < \infty$ (the case $p = 1$ is excluded by the assumption $r/d > 1/p$). Here we start with the case of Bessel potential spaces. We put $r_1 = \lceil r \rceil$. Setting $s_0 = \lceil s \rceil - 1$, $s_1 = \lfloor s \rfloor + 1$, we have $r_1 \ge s_1 > s_0$, and by (42),

$$\|I - P_{l,0} : W_p^{r_1}(Q) \to W_p^{s_i}(Q)\| \;\le\; c\, 2^{-(r_1 - s_i)l} \quad (i = 0, 1). \tag{76}$$

Choosing $0 < \theta < 1$ in such a way that $s = (1-\theta)s_0 + \theta s_1$ and using (47) and (52), complex interpolation of (76) gives

$$\|I - P_{l,0} : W_p^{r_1}(Q) \to H_p^s(Q)\| \;\le\; c\, 2^{-(r_1 - s)l}. \tag{77}$$

In view of (47), this proves the case $r = r_1 = \lceil r \rceil$. Now we assume $r < r_1$ and choose $r_0 \in \mathbb{R}$, $0 < r_0 < r$ sufficiently small such that

$$r - s > \frac{r_1 - r}{r_1 - r_0} r_0. \tag{78}$$

Then put

$$\theta = \frac{r - r_0}{r_1 - r_0}, \quad s_0 = 0 \quad s_1 = \frac{s}{\theta}, \quad p_0 = \frac{rp}{r_0}, \quad p_1 = \frac{rp}{r_1}. \tag{79}$$

It is readily checked that

$$r = (1-\theta)r_0 + \theta r_1, \quad s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{80}$$

Furthermore, since $r/d > 1/p$, it follows that

$$\frac{r_0}{d} > \frac{1}{p_0}, \quad \frac{r_1}{d} > \frac{1}{p_1},$$

and, because of $r_0 < r_1 \le d$, this together with (79) implies $1 < p_1 < p_0 < \infty$. Finally, (78) and (80) give

$$\theta r_1 = r - (1-\theta)r_0 = r - \frac{r_1 - r}{r_1 - r_0} r_0 > s = \theta s_1,$$

so $r_1 > s_1$. From (77) we conclude

$$\| I - P_{l,0} : W_{p_1}^{r_1}(Q) \to H_{p_1}^{s_1}(Q) \| \le c\, 2^{-(r_1 - s_1)l}.$$

By the already shown case of $s = s_0 = 0$ of (68) we have

$$\| I - P_{l,0} : W_{p_0}^{r_0}(Q) \to H_{p_0}^{s_0}(Q) \| \le c\, 2^{-(r_0 - s_0)l}.$$

Complex interpolation together with (47), (52), and (80) gives

$$\| I - P_{l,0} : H_p^r(Q) \to H_p^s(Q) \| \le c\, 2^{-(r-s)l}.$$

It remains to consider the case of Besov spaces for $r \le d$ and $1 < p < \infty$. But this can be derived from the already completed case of Bessel potential spaces as follows. We choose $s_i, r_i \in \mathbb{R}$ $(i = 0, 1)$, close to s and r, respectively, in such a way that

$$0 < s_0 < s < s_1 < r_0 < r < r_1, \quad \varrho > r_1, \quad \sigma > r_1 + 1. \tag{81}$$

By (68),

$$\| I - P_{l,0} : H_p^{r_i}(Q) \to H_p^{s_j}(Q) \| \le c\, 2^{-(r_i - s_j)l} \quad (i, j \in \{0, 1\}). \tag{82}$$

Using (50), (51), and (81), real interpolation of (82) gives for $1 \le u \le \infty$

$$\|I - P_{l,0} : B^r_{pu}(Q) \to H^{s_j}_p(Q)\| \le c\, 2^{-(r-s_j)l} \quad (j = 0, 1). \tag{83}$$

Interpolating (83) in a respective way yields for $1 \le v \le \infty$

$$\|I - P_{l,0} : B^r_{pu}(Q) \to B^s_{pv}(Q)\| \le c\, 2^{-(r-s)l}.$$

$\square$

Remark. In accordance with the cited literature, in this chapter we considered only spaces of complex-valued functions. The statements of the propositions and theorems remain valid, however, also for the case of real-valued functions. This can be derived in a formal way from the complex case using the following facts. For the involved function spaces X we have

$$f \in X \quad \text{iff} \quad \operatorname{Re} f, \operatorname{Im} f \in X$$

and

$$c_1(\|\operatorname{Re} f\|_X + \|\operatorname{Im} f\|_X) \le \|f\|_X \le c_2(\|\operatorname{Re} f\|_X + \|\operatorname{Im} f\|_X),$$

which is a consequence of [19], Th. 1.118. Moreover, the involved approximating operators map real-valued functions to real-valued functions, and finally, the lower bound arguments can be based on real functions.

6 An application to elliptic PDE

The results obtained above have some direct consequences for the information complexity of solution of elliptic partial differential equations (see [6], [7], and the references in there, for previous results in this direction). Let $d, m \in \mathbb{N}$, $d \ge 2$, let $Q \subset \mathbb{R}^d$ be a bounded C^∞ domain (see, e.g., [15] for the definition). We consider the homogeneous boundary value problem

$$\mathscr{L} z(x) = f(x) \quad (x \in Q^0) \tag{84}$$

$$\mathscr{B}_j z(x) = 0 \quad (x \in \partial Q,\ j = 1, \dots, m), \tag{85}$$

where $\mathscr{L}$ is a differential operator of order $2m$ on Q, that is

$$\mathscr{L} z = \sum_{|\alpha| \le 2m} a_\alpha(x) D^\alpha z(x),$$

and the $\mathscr{B}_j$ are boundary operators

$$\mathscr{B}_j z = \sum_{|\alpha| \le m_j} b_{j\alpha}(x) D^\alpha z(x), \tag{86}$$

where $m_j \le 2m-1$ and $a_\alpha \in C^\infty(Q)$ and $b_{j\alpha} \in C^\infty(\partial Q)$ are complex-valued infinitely differentiable functions.

We asssume that $(\mathscr{L}, \{\mathscr{B}_j\})$ is regular elliptic ([15], 5.2.1/4), and that 0 is not in the spectrum of $\mathscr{L}$. By [15], Theorem 5.5.1(b) it follows that $\mathscr{L}$ is an isomorphism from $W^{s+2m}_{q,\{\mathscr{B}_j\}}(Q)$ (the subspace of $W^{s+2m}_q(Q)$ consisting of those functions which satisfy (85)) to $W^s_q(Q)$ for $s \in \mathbb{N}_0$ and $1 < q < \infty$. We define the solution operator $S = \mathscr{L}^{-1}J$ of the elliptic problem (84–85) as follows:

$$S : W^r_p(Q) \xrightarrow{J} W^s_q(Q) \xrightarrow{\mathscr{L}^{-1}} W^{s+2m}_q(Q).$$

This means, we seek to approximate the full solution u, for right-hand sides f from $W^r_p(Q)$. Note that we consider S as an operator into $W^{s+2m}_q(Q)$. In particular, the error is measured in the norm of $W^{s+2m}_q(Q)$.

Corollary 6.1. *Let $d \in \mathbb{N}$, $d \ge 2$, $r, s \in \mathbb{N}_0$, $1 \le p \le \infty$, $1 < q < \infty$, and assume that $(r-s)/d \ge \max(1/p - 1/q, 0)$. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ the following hold:*

$$c_1 n^{-\gamma} \le e_n^{\mathrm{ran}}(S, \mathcal{B}_{W^r_p(Q)}, W^{s+2m}_q(Q)) \le c_2 n^{-\gamma}$$

with

$$\gamma = \frac{r-s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+ .$$

If, in addition $W^r_p(Q)$ is embedded into $C(\bar{Q})$, that is, (41) holds, we also have

$$c_1 n^{-\gamma} \le e_n^{\mathrm{det}}(S, \mathcal{B}_{W^r_p(Q)}, W^{s+2m}_q(Q)) \le c_2 n^{-\gamma}.$$

If (41) does not hold, then

$$c_1 \le e_n^{\mathrm{det}}(S, \mathcal{B}_{W^r_p(Q)}, W^{s+2m}_q(Q)) \le c_2.$$

Proof. The upper bounds are direct consequences of the above mentioned isomorphism property of $\mathscr{L}$ and Theorems 3.1, 4.2, and 4.3. For the lower bound we additionally remark that by [15], Theorem 5.5.2(b), $W^{s+2m}_{q,\{\mathscr{B}_j\}}(Q)$ is a complemented subspace of $W^{s+2m}_q(Q)$, hence, up to a constant factor, algorithms with values in $W^{s+2m}_q(Q)$ cannot give a smaller error than those with values in $W^{s+2m}_{q,\{\mathscr{B}_j\}}(Q)$. Therefore, also the lower bounds follow from the isomorphism property and Theorems 3.1, 4.2, and 4.3.

□

References

[1] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[3] D. E. Edmunds, H. Triebel, Function Spaces, Entropy Numbers, Differential Operators, Cambridge Univ. Press, Cambridge, 1996.

[4] S. Heinrich, Random approximation in numerical analysis, in: K. D. Bierstedt, A. Pietsch, W. M. Ruess, D. Vogt (Eds.), Functional Analysis, Marcel Dekker, New York, 1993, 123 – 171.

[5] S. Heinrich, Monte Carlo approximation of weakly singular integral operators, J. Complexity 22 (2006), 192–219.

[6] S. Heinrich, The randomized information complexity of elliptic PDE, J. Complexity 22 (2006), 220–249.

[7] S. Heinrich, Randomized approximation of Sobolev embeddings, in: Monte Carlo and Quasi-Monte Carlo Methods 2006 (A. Keller, S. Heinrich, H. Niederreiter, eds.), Springer, Berlin, 2008, 445 – 459.

[8] S. Heinrich, Randomized approximation of Sobolev embeddings III (submitted).

[9] P. Mathé, Random approximation of Sobolev embeddings. J. Complexity 7 (1991), 261–281.

[10] E. Novak, Deterministic and Stochastic Error Bounds in Numerical Analysis, Lecture Notes in Mathematics 1349, Springer-Verlag, Berlin, 1988.

[11] E. Novak, H. Triebel, Function spaces in Lipschitz domains and optimal rates of convergence for sampling, Constr. Approx. 23 (2006), 325–350.

[12] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume 1, Linear Information, European Math. Soc., Zürich, 2008.

[13] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.

[14] J. F. Traub, G. W. Wasilkowski, and H. Woźniakowski, Information-Based Complexity, Academic Press, 1988.

[15] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, Deutscher Verlag des Wissenschaften, Berlin, and North-Holland Publishing Company, Amsterdam - New York - Oxford, 1978.

[16] H. Triebel, Theory of Function Spaces, Birkhäuser, Basel, 1983.

[17] H. Triebel, Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers, Rev. Mat. Complut. 15, No. 2 (2002), 475–524.

[18] H. Triebel, Sampling numbers and embedding constants, Proc. Steklov Inst. Math. 248 (2005), 268–277.

[19] H. Triebel, Theory of Function Spaces III, Birkhäuser, Basel, 2006.

[20] J. Vybíral, Sampling numbers and function spaces, J. Complexity 23 (2007), 773-792.

[21] J. Vybíral, Dilation operators and sampling numbers, J. Funct. Spaces Appl. 6 (2008), 17-46

[22] G. W. Wasilkowski, Randomization for continuous problems, J. Complexity 5 (1989), 195–218.