The randomized complexity of initial value problems

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Abstract

We study the complexity of randomized solution of initial value problems for systems of ordinary differential equations (ODE). The input data are assumed to be γ-smooth ($\gamma = r + \varrho$: the r-th derivatives satisfy a ϱ-Hölder condition). Recently, the following almost sharp estimate of the order of the n-th minimal error was given by Kacewicz (Almost optimal solution of initial-value problems by randomized and quantum algorithms, J. Complexity 22 (2006), 676-690):

$$c_1 n^{-\gamma-1/2} \le e_n^{\rm ran} \le c_2(\varepsilon) n^{-\gamma-1/2+\varepsilon},$$

with an arbitrary $\varepsilon > 0$. We present a Taylor Monte Carlo method and show that it has error rate $n^{-\gamma-1/2}$, this way establishing the exact order of the randomized n-th minimal error.

1 Introduction

We consider the numerical solution of initial value problems for systems of ODE

$$y'(x) = f(x, y(x)) \quad (x \in [a, b]), \tag{1}$$
$$y(a) = y_0, \tag{2}$$

with $y_0 \in \mathbb{R}^d$ and $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ (precise assumptions on f will be given below). Randomized algorithms for such problems where first considered by Stengle [14, 15]. In [14] a general family of such algorithms was introduced,

while [15] contains a convergence analysis for a specific member of that family (a method of maximal order 2). Quasi-Monte Carlo methods of similar nature were introduced and studied by Coulibaly and Lécot [2, 12].

In the framework of information-based complexity (IBC) the randomized solution of initial value problems for systems of ODE was first studied by Kacewicz in [9, 10, 11]. He seeks to find optimal algorithms for input data of an arbitrary, fixed degree of smoothness (and thus, methods of arbitrary high order have to be considered). For this purpose on each interval of a uniform partition of $[a, b]$ an integral is treated by a suitable Monte Carlo method with variance reduction. Iterative refinement then leads to an order of convergence which matches the lower bound up to an arbitrary $\varepsilon > 0$ in the exponent (see [11] and relation (8) below). The precise order of the n-th minimal error, however, remained open.

In this paper we solve this problem. Moreover, we show that a simple direct approach, namely taking the uniform grid and for each interval just one random sample for a suitable integral leads to the optimal order. Our variance reduction is based on the Taylor method like that in [11], however, a simpler control variate is used. The convergence analysis uses certain martingale inequalities, similar to [15]. In addition to the stochastic convergence rate we show that the proposed method is also optimal in the deterministic setting, meaning that the optimal order is obtained for each fixed realization of the involved random variables (with constants not depending on the realization).

We consider a somewhat larger family of function classes than done in [9, 10, 11], including those with no or small (Hölder) smoothness in x (time) and just the Lipschitz condition in y (space). Our results imply that for such classes deterministic solution can be arbitrarily hard, up to intractable (no smoothness), while randomized algorithms still provide a convergence rate of $n^{-1/2}$.

The paper is organized as follows. In Section 2 we introduce the precise problem formulation, describe basics of the framework of IBC, and state the main result. Section 3 contains the algorithm, its analysis and, based on this, the proof of the main result. In section 4 we give some further comments and results, including an estimate of the error distribution and the comparison between deterministic and randomized setting.

2 Preliminaries

Let $d \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$. For $-\infty < a < b < +\infty$, $\kappa, L > 0$, $r \in \mathbb{N}_0$, $0 \le \varrho \le 1$ we consider the following class $C_d^{r,\varrho}(a, b, \kappa, L)$ of functions $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ having continuous partial derivatives D^α with $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^{d+1}$ of order $|\alpha| = \alpha_0 + \alpha_1 \cdots + \alpha_d \le r$

$$D^\alpha f(x, z) = \frac{\partial^{|\alpha|} f(x, z)}{\partial x^{\alpha_0} \partial z_1^{\alpha_1} \dots \partial z_d^{\alpha_d}}$$

satisfying

$$\begin{aligned} |D^\alpha f(x, z)| &\le \kappa \qquad (|\alpha| \le r), && (3)\\ |D^\alpha f(x, z) - D^\alpha f(t, v)| &\le \kappa(|x - t|^\varrho + |z - v|^\varrho) \quad (|\alpha| = r), && (4)\\ |f(x, z) - f(x, v)| &\le L|z - v| && (5) \end{aligned}$$

for $x, t \in [a, b]$, $z, v \in \mathbb{R}^d$. We denote the total degree of smoothness by

$$\gamma = r + \varrho.$$

For example, $C_d^{0,0}(a, b, L, \kappa)$ consists of bounded functions just continuous in x and satisfying a Lipschitz condition with respect to y. The $C_d^{r,\varrho}$ are also the classes considered in [11] (up to equivalence in the sense of being contained in scalar multiples of each other), except that in [11] $\varrho > 0$ and $\gamma \ge 1$ were assumed. This latter restriction results from the reduction to autonomous systems and the Lipschitz condition imposed upon them. It turns out though that the cases $C_d^{0,\varrho}$ with $0 \le \varrho < 1$ are particularly interesting for the comparison between deterministic and randomized setting, see the comments in section 4.

We work in the setting of IBC, as discussed in [16, 13]. For the precise notions used here we refer to [4, 5]. An abstract numerical problem is described by a tuple $\mathcal{P} = (F, G, S, K, \Lambda)$. The set F is the set of input data, in our case

$$F = \left\{ (f, y_0) : \, f \in C_d^{r,\varrho}(a, b, \kappa, L), \, y_0 \in \mathbb{R}^d, \, |y_0| \le \sigma \right\}, \qquad (6)$$

where $\sigma > 0$ is any fixed number, G is a normed linear space and $S : F \to G$ an (in general nonlinear) operator, the solution operator, which maps the input $\psi \in F$ to the exact solution $S(\psi)$. In our case we put $G = B([a, b], \mathbb{R}^d)$,

the space of all $\mathbb{R}^d$-valued, bounded on $[a,b]$ functions, equipped with the supremum norm

$$\|g\|_\infty = \sup_{x\in[a,b]} |g(x)|.$$

Then for $(f,y_0) \in F$ we let $S(f,y_0) = y$ be the solution of the initial value problem (1–2). The choice of G means that we measure the error in the uniform norm. The set K is the scalar field $\mathbb{R}$, and Λ is the class of admissible information functionals, that is, Λ is a set of mappings from F to $K = \mathbb{R}$. Here we put

$$\Lambda = \{\delta_{i,s}^\alpha : \ 1 \le i \le d, s \in [a,b]\times\mathbb{R}^d, \alpha \in \mathbb{N}_0^{d+1}, |\alpha| \le r\} \cup \{\delta_i : \ 1 \le i \le d\},$$

where

$$\begin{aligned} \delta_{i,s}^\alpha(f,y_0) &= D^\alpha f_i(s), \\ \delta_i(f,y_0) &= y_{0,i}, \end{aligned}$$

and f_i and $y_{0,i}$ are the i-th components of f and y_0, respectively. Hence the admissible information consists of values of the components of f and their derivatives (and, of course, the initial values).

The precise notion of an abstract adaptive randomized algorithm is technically somewhat involved, so here we just sketch some basic features and refer to [4, 5] for details. A randomized algorithm for the solution of $\mathcal{P}$ is a family $A = (A_\omega)_{\omega\in\Omega}$, where $(\Omega,\Sigma,\mathbb{P})$ is the underlying probability space and each A_ω is a mapping $A_\omega : F \to G$. With ω – the parameter incorporating all randomness of the algorithm – being fixed, $A_\omega : F \to G$ is a deterministic abstract algorithm, that is, stands for a deterministic procedure depending on ω which uses values of information functionals on (f,y_0) in an adaptive way to produce an approximation $A_\omega(f,y_0)$ to $S(f,y_0) = y$. The error of A is defined as

$$e(S,A,F) = \sup_{(f,y_0)\in F} \left(\mathbb{E}\,\|S(f,y_0) - A_\omega(f,y_0)\|_\infty^2\right)^{1/2}.$$

Another important quantity related to an algorithm A is the cardinality $\mathrm{card}(A,F)$. Let $\mathrm{card}(A_\omega,f,y_0)$ be the number of information functionals called in the course of computation of $A_\omega(f,y_0)$. Then we put

$$\mathrm{card}(A,F) = \sup_{(f,y_0)\in F} \left(\mathbb{E}\,\mathrm{card}(A_\omega,f,y_0)^2\right)^{1/2}.$$

The quantity $\mathrm{card}(A,F)$ will be our cost measure. Although this notion of cost neglects arithmetic operations, we shall see that the order optimal

algorithm studied later has a number of arithmetic operations proportional to the cardinality.

The crucial quantity of IBC, the n-th minimal error, is defined for $n \in \mathbb{N}_0$ as

$$e_n^{\mathrm{ran}}(S,F) = \inf_{\mathrm{card}(A,F)\le n} e(S,A,F).$$

That is, $e_n^{\mathrm{ran}}(S,F)$ is the minimal possible error among all randomized algorithms that use (on the average) at most n information functionals.

The deterministic setting can be viewed as a special case of the above by admitting only trivial one-point probability spaces $\Omega = \{\omega_0\}$ (meaning that there is no dependence on randomness). We use the notation $e_n^{\mathrm{det}}(S,F)$ for the n-th minimal error in this setting.

Now let us recall previous results on the complexity of the initial value problem (1–2), due to Kacewicz. For the deterministic setting the following holds: There are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$

$$c_1 n^{-\gamma} \le e_n^{\mathrm{det}}(S,F) \le c_2 n^{-\gamma}. \tag{7}$$

This is essentially shown in [6, 7, 8], see also the comments on p. 827 of [9] and those after Proposition 3 of the present paper. For the randomized setting it is proved in [11] that there is a constant $c_1 > 0$ and for each $\varepsilon > 0$ a constant $c_2(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$

$$c_1 n^{-\gamma-1/2} \le e_n^{\mathrm{ran}}(S,F) \le c_2(\varepsilon) n^{-\gamma-1/2+\varepsilon}. \tag{8}$$

In this paper we show that the lower bound of (8) is sharp. The main result is the following

Theorem 1. *Let $r \in \mathbb{N}_0$, $0 \le \varrho \le 1$, $a < b$, $\kappa, L, \sigma > 0$, let F be defined by (3–6) and let $\gamma = r + \varrho$. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ the n-th minimal error of the initial value problem (1–2) satisfies*

$$c_1 n^{-\gamma-1/2} \le e_n^{\mathrm{ran}}(S,F) \le c_2 n^{-\gamma-1/2}. \tag{9}$$

3 The algorithm and its analysis

We fix $n \in \mathbb{N}$, $n \ge 2$, put $h = (b-a)/n$, $x_k = a + kh$ $(k = 0, 1, \ldots, n)$ and let ξ_k $(k = 1, \ldots, n-1)$ be independent random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$, with $\xi_k(\omega) \in [x_{k-1}, x_k]$ for all $\omega \in \Omega$ and ξ_k being uniformly distributed on $[x_{k-1}, x_k]$. Inductively define $y_k \in \mathbb{R}^d$ for $k = 1, \ldots, n-1$ as follows. Let $0 \le k < n-1$ and suppose that y_k is already defined (y_0 is the

initial value). Let $u_k(x)$ ($x \in [x_k, x_{k+1}]$) be the solution of the local initial value problem

$$\begin{aligned} u_k'(x) &= f(x, u_k(x)) \quad (x \in [x_k, x_{k+1}]), && (10) \\ u_k(x_k) &= y_k. && (11) \end{aligned}$$

The smoothness of f implies that u_k is $(r+1)$-times continuously differentiable and for $j = 2, \ldots, r+1$ and $x \in [x_k, x_{k+1}]$

$$u_k^{(j)}(x) = \left(\frac{\partial}{\partial x} + \sum_{i=1}^{d} f_i(x,z) \frac{\partial}{\partial z_i} \right)^{j-1} f(x,z) \Bigg|_{z=u_k(x)}. \qquad (12)$$

Let

$$p_k(x) = \sum_{j=0}^{r+1} \frac{u_k^{(j)}(x_k)}{j!} (x - x_k)^j \quad (x \in [x_k, x_{k+1}]) \qquad (13)$$

be the Taylor polynomial of degree $r+1$ of $u_k(x)$ in the point x_k. By (10) and (12), the values $u_k^{(j)}(x_k)$ needed for the coefficients of p_k in (13) can be expressed by values of f and its partial derivatives up to order r at the point (x_k, y_k). Now we put

$$y_{k+1} = p_k(x_{k+1}) + h \left(f(\xi_{k+1}, p_k(\xi_{k+1})) - p_k'(\xi_{k+1}) \right). \qquad (14)$$

Having determined the y_k, the full approximate solution $\bar{y}(x)$ on $[a, b]$ is now defined as

$$\bar{y}(x) = \begin{cases} p_k(x) & \text{if} \quad x \in [x_k, x_{k+1}) \quad \text{and} \quad 0 \le k < n-1, \\ p_{n-1}(x) & \text{if} \quad x \in [x_{n-1}, x_n]. \end{cases} \qquad (15)$$

While the choice (15) is clear, let us give some motivating explanations for (14): In a first step we approximate

$$\begin{aligned} y(x_{k+1}) &= y(x_k) + \int_{x_k}^{x_{k+1}} f(t, y(t)) dt \\ &\approx y_k + \int_{x_k}^{x_{k+1}} f(t, p_k(t)) dt. && (16) \end{aligned}$$

This is the (order optimal) approximation from [8] in the setting where arbitrary linear information is admissible. In the setting considered here only values of functions and derivatives are allowed. Thus, we approximate the last integral. This is done by a variance reduced Monte Carlo method

with one sample. Namely, we use $p_k'(t)$ as a control variate for $f(t, p_k(t))$. Indeed, $p_k'(t)$ is an approximation to $u_k'(t) = f(t, u_k(t))$, which, in turn, is close to $f(t, p_k(t))$ (the rates of approximation of these quantities are analysed below). Thus,

$$\begin{aligned}
&\int_{x_k}^{x_{k+1}} f(t, p_k(t))dt \\
&= \int_{x_k}^{x_{k+1}} p_k'(t)dt + \int_{x_k}^{x_{k+1}} (f(t, p_k(t)) - p_k'(t))dt \\
&\approx p_k(x_{k+1}) - p_k(x_k) + h\left(f(\xi_{k+1}, p_k(\xi_{k+1})) - p_k'(\xi_{k+1})\right). \qquad (17)
\end{aligned}$$

We insert (17) into (16). Since $p_k(x_k) = y_k$, this leads to (14).

Now we estimate the error of this algorithm. Note that throughout the paper $c, c_1, c_2, \ldots$ stand for constants, depending only on the problem parameters $d, r, \varrho, a, b, L, \kappa, \sigma$, but neither on the quantitative algorithm parameters n, k, etc. nor on the particular input (f, y_0). Moreover, the same symbol may denote different constants, even in a sequence of relations.

Proposition 1. *There is a constant $c > 0$ such that for all $(f, y_0) \in F$ and for all $n \in \mathbb{N}$ with $n \geq 2$ the error of the algorithm described above satisfies*

$$\left(\mathbb{E}\,\|y - \bar{y}\|_\infty^2\right)^{1/2} \leq cn^{-\gamma-1/2}. \qquad (18)$$

Proof. We first note the following smoothness properties of the u_k: There are constants $c_1, c_2 > 0$ such that for all $(f, y_0) \in F$, n, and k, the following hold:

$$|u_k^{(j)}(x)| \leq c_1 \qquad (x \in [x_k, x_{k+1}],\, 0 \leq j \leq r+1), \qquad (19)$$

$$|u_k^{(r+1)}(x) - u_k^{(r+1)}(t)| \leq c_2|x-t|^\varrho \qquad (x, t \in [x_k, x_{k+1}]). \qquad (20)$$

Indeed, for $j \geq 1$ relation (19) is an immediate consequence of (3), (10), and (12). This, in turn, together with (13) and (14) implies $|y_k| \leq c$, and hence (19) also for $j = 0$. To see (20), we observe that, using (10) and (12), each component of the function $u_k^{(r+1)}(x)$ can be expressed as a sum of M products of the form

$$\prod_{l=1}^{m} \left(D^{\beta_l} f_{i_l}\right)(x, u_k(x))$$

with

$$m \leq r+1, \quad \beta_l \in \mathbb{N}_0^{d+1}, \quad 1 \leq i_l \leq d, \quad \sum_{l=1}^{m} |\beta_l| = r,$$

and M depending only on r and d. Now (20) follows from (4) and (19). By the Taylor series with integral remainder term, we have

$$u_k(x) = p_k(x) + \frac{1}{r!}\int_{x_k}^{x}(x-t)^r\left(u_k^{(r+1)}(t) - u_k^{(r+1)}(x_k)\right)dt. \tag{21}$$

We denote

$$\mu_k = \sup_{x\in[x_k,x_{k+1}]}|u_k(x) - p_k(x)|. \tag{22}$$

This is the deviation of the approximate solution $\bar{y}_k$ given by (15) from the solution (10–11) of the local problem on $[x_k, x_{k+1}]$. From (20) and (21) we obtain

$$\mu_k \le ch^{r+1+\varrho} = ch^{\gamma+1}. \tag{23}$$

Differentiating (21) we get for $r \ge 1$

$$u_k'(x) = p_k'(x) + \frac{1}{(r-1)!}\int_{x_k}^{x}(x-t)^{r-1}\left(u_k^{(r+1)}(t) - u_k^{(r+1)}(x_k)\right)dt, \tag{24}$$

which gives, using (20) again,

$$\sup_{x\in[x_k,x_{k+1}]}|u_k'(x) - p_k'(x)| \le ch^{\gamma}. \tag{25}$$

This also holds for $r = 0$ (hence $\gamma = \varrho$) since in this case $p_k'(x) \equiv u_k'(x_k)$, and we can apply (20) directly. Using (5), (10), (22), (23), and (25) we get

$$\begin{aligned}
&\sup_{x\in[x_k,x_{k+1}]}|f(x,p_k(x)) - p_k'(x)| \\
\le\ &\sup_{x\in[x_k,x_{k+1}]}\big(|f(x,p_k(x)) - f(x,u_k(x))| + |f(x,u_k(x)) - p_k'(x)|\big) \\
\le\ &L\mu_k + \sup_{x\in[x_k,x_{k+1}]}|u_k'(x) - p_k'(x)| \le ch^{\gamma}
\end{aligned} \tag{26}$$

(this explains why we use $p_k'(x)$ as a control variate to $f(x,p_k(x))$). Finally we observe that Gronwall's lemma yields

$$\sup_{x\in[x_k,x_{k+1}]}|y(x) - u_k(x)| \le e^{Lh}|y(x_k) - y_k|. \tag{27}$$

Let $\mathbb{E}_k$ denote the conditional expectation with respect to the σ-algebra $\mathcal{A}_k \subseteq \Sigma$ generated by $\{\xi_1, \ldots, \xi_k\}$ and let $\mathbb{E}_0 = \mathbb{E}$ be just the expectation. Then we get from (14)

$$\mathbb{E}_k y_{k+1} = y_k + \int_{x_k}^{x_{k+1}} f(t, p_k(t))dt. \tag{28}$$

We denote the error in the point x_k by

$$e_k = y(x_k) - y_k$$

and split it at step $k+1$ as

$$e_{k+1} = e_k + g_{k+1} + d_{k+1} + \eta_{k+1} \tag{29}$$

with

$$g_{k+1} = \int_{x_k}^{x_{k+1}} (f(t, y(t)) - f(t, u_k(t)))\, dt, \tag{30}$$

$$d_{k+1} = \int_{x_k}^{x_{k+1}} (f(t, u_k(t)) - f(t, p_k(t)))\, dt, \tag{31}$$

and

$$\eta_{k+1} = y_k + \int_{x_k}^{x_{k+1}} f(t, p_k(t))dt - y_{k+1}. \tag{32}$$

Let us mention the meaning of the error terms in (29). We have

$$g_{k+1} = y(x_{k+1}) - u_k(x_{k+1}) - (y(x_k) - y_k),$$

so g_k is the change due to propagation of the error from x_k to x_{k+1} along the trajectories given by the differential equations (1–2) and (10–11), while

$$d_{k+1} + \eta_{k+1} = u_k(x_{k+1}) - y_{k+1}$$

is the local error. Moreover, by (28),

$$\begin{aligned} d_{k+1} &= u_k(x_{k+1}) - \mathbb{E}_k y_{k+1}, \\ \eta_{k+1} &= \mathbb{E}_k y_{k+1} - y_{k+1}, \end{aligned} \tag{33}$$

hence d_{k+1} is the conditional mean of the local error, and η_{k+1} its fluctuation around the mean.

Next we estimate these error terms. From (22), (23), (27), and Lipschitz continuity (5) we get

$$|g_{k+1}| \leq ch|e_k|, \tag{34}$$

$$|d_{k+1}| \leq ch^{\gamma+2}. \tag{35}$$

Furthermore, (14) gives

$$\begin{aligned}\eta_{k+1} &= y_k + \int_{x_k}^{x_{k+1}} f(t, p_k(t))dt \\ &\quad - \big(p_k(x_{k+1}) + h(f(\xi_{k+1}, p_k(\xi_{k+1})) - p_k'(\xi_{k+1}))\big) \\ &= \int_{x_k}^{x_{k+1}} \big(f(t, p_k(t)) - p_k'(t)\big)\, dt \\ &\quad -h(f(\xi_{k+1}, p_k(\xi_{k+1})) - p_k'(\xi_{k+1})). \end{aligned} \tag{36}$$

Therefore (26) implies

$$|\eta_{k+1}| \le ch^{\gamma+1}. \tag{37}$$

From (29) and $e_0 = 0$ we get

$$e_k = \sum_{j=1}^{k} (g_j + d_j + \eta_j).$$

This together with (34) gives for $1 \le k \le n-1$

$$\begin{aligned}\max_{0\le j\le k} |e_j| &\le \sum_{j=1}^{k} |g_j| + \sum_{j=1}^{k} |d_j| + \max_{1\le j\le k} \Big|\sum_{i=1}^{j} \eta_i\Big| \\ &\le ch\sum_{j=0}^{k-1} |e_j| + \sum_{j=1}^{k} |d_j| + \max_{1\le j\le k} \Big|\sum_{i=1}^{j} \eta_i\Big| \\ &\le ch\sum_{j=0}^{k-1} \max_{0\le i\le j} |e_i| + \sum_{j=1}^{k} |d_j| + \max_{1\le j\le k} \Big|\sum_{i=1}^{j} \eta_i\Big|. \end{aligned} \tag{38}$$

We introduce the function $v(x)$ for $x \in [a, b]$ by

$$v(x) = \begin{cases} \max_{0\le j\le k} |e_j| & \text{if} \quad x \in [x_k, x_{k+1}) \quad \text{and} \quad 0 \le k < n-1, \\ \max_{0\le j\le n-1} |e_j| & \text{if} \quad x \in [x_{n-1}, x_n]. \end{cases}$$

It follows from (38) that for $x \in [a, b]$

$$v(x) \le c\int_a^x v(t)dt + \sum_{j=1}^{n-1} |d_j| + \max_{1\le k\le n-1} \Big|\sum_{i=1}^{k} \eta_i\Big|.$$

From Gronwall's lemma we conclude

$$\begin{aligned}\max_{0\le k\le n-1}|e_k| &= v(b)\le e^{c(b-a)}\left(\sum_{j=1}^{n-1}|d_j|+\max_{1\le k\le n-1}\Big|\sum_{i=1}^{k}\eta_i\Big|\right)\\ &\le c\sum_{j=1}^{n-1}|d_j|+c\max_{1\le k\le n-1}\Big|\sum_{i=1}^{k}\eta_i\Big|. \end{aligned}\tag{39}$$

Using (22), (27), and (39) we get

$$\begin{aligned}\|y-\bar y\|_\infty &= \max_{0\le k\le n-1}\sup_{x\in[x_k,x_{k+1}]}|y(x)-p_k(x)|\\ &\le \max_{0\le k\le n-1}\sup_{x\in[x_k,x_{k+1}]}(|y(x)-u_k(x)|+|u_k(x)-p_k(x)|)\\ &\le \max_{0\le k\le n-1}(c|e_k|+\mu_k)\\ &\le c\sum_{j=1}^{n-1}|d_j|+c\max_{1\le k\le n-1}\Big|\sum_{i=1}^{k}\eta_i\Big|+\max_{0\le k\le n-1}\mu_k. \end{aligned}\tag{40}$$

Consequently,

$$\begin{aligned}&\mathbb{E}\,\|y-\bar y\|_\infty^2\\ &\le c\,\mathbb{E}\Big(\sum_{j=1}^{n-1}|d_j|\Big)^2+c\,\mathbb{E}\max_{1\le k\le n-1}\Big|\sum_{i=1}^{k}\eta_i\Big|^2+2\,\mathbb{E}\max_{0\le k\le n-1}\mu_k^2. \end{aligned}\tag{41}$$

By (33) the sequence of random variables $\Big(\sum_{i=1}^{k}\eta_i\Big)_{k=1}^{n-1}$ is a martingale, and by convexity of the norm $|\cdot|$,

$$\left(\Big|\sum_{i=1}^{k}\eta_i\Big|\right)_{k=1}^{n-1}$$

is a non-negative submartingale. From Doob's inequality ([3], Ch. VII, Th. 3.4) we obtain

$$\mathbb{E}\max_{1\le k\le n-1}\Big|\sum_{i=1}^{k}\eta_i\Big|^2\le 4\mathbb{E}\Big|\sum_{i=1}^{n-1}\eta_i\Big|^2=4\sum_{i=1}^{n-1}\mathbb{E}\,|\eta_i|^2.\tag{42}$$

Combining this with (41) yields

$$\begin{aligned}&\mathbb{E}\,\|y-\bar y\|_\infty^2\\ &\le c\,\mathbb{E}\Big(\sum_{j=1}^{n-1}|d_j|\Big)^2+c\sum_{i=1}^{n-1}\mathbb{E}\,|\eta_i|^2+2\,\mathbb{E}\max_{0\le k\le n-1}\mu_k^2. \end{aligned}\tag{43}$$

It remains to insert the estimates (35), (37), and (23) to obtain

$$\mathbb{E}\,\|y-\bar{y}\|_\infty^2 \le ch^{2\gamma+1}, \tag{44}$$

which completes the proof of Proposition 1. □

Proof of Theorem 1. The upper bound follows directly from Proposition 1 and the obvious fact that the algorithm needs not more than cn values of function f and its derivatives. Formally, this covers the case $n \ge c_0$ for some $c_0 > 0$. For $n < c_0$ it suffices to use the trivial zero algorithm $A_\omega(f, y_0) \equiv 0$ since, as easily seen, for all $(f, y_0) \in F$ the solution $y = S(f, y_0)$ satisfies $\|y\|_\infty \le \sigma + \kappa(b-a)$.

Concerning the lower bound, it clearly suffices to consider the case $d = 1$. We shall reduce an integration problem to the solution of initial value problems. Let

$$F_0 = C_0^{r,\varrho}(a, b, \kappa)$$

be the set of all functions $f : [a, b] \to \mathbb{R}$ satisfying for $x, t \in [a, b]$

$$|f^{(j)}(x)| \le \kappa \qquad (0 \le j \le r), \tag{45}$$

$$|f^{(r)}(x) - f^{(r)}f(t)| \le \kappa|x-t|^\varrho, \tag{46}$$

let $G_0 = \mathbb{R}$ and define $S_0 : F_0 \to G_0$ by

$$S_0(f) = \int_a^b f(t)dt.$$

Moreover, let $K_0 = \mathbb{R}$ and

$$\Lambda_0 = \{\delta_s^j : \ s \in [a, b], 0 \le j \le r\}$$

with $\delta_s^j(f) = f^{(j)}(s)$. Finally, let

$$R : F_0 \to F = C_1^{r,\varrho}(a, b, \kappa, L) \times [-\sigma, \sigma],$$

be defined by

$$R(f) = (\bar{f}, 0)$$

where $\bar{f}$ is just f, considered as a function of two variables x, z, with no dependence on z, and let $\Psi : B([a, b], \mathbb{R}) \to \mathbb{R}$ be given by

$$\Psi(g) = g(b).$$

Obviously,

$$S_0 = \Psi \circ S \circ R.$$

This shows that the integration problem $\mathcal{P}_0 = (F_0, G_0, S_0, K_0, \Lambda_0)$ reduces to $\mathcal{P}$ (see [5] for the formal definition and additional details like the requirements on R, which are easily seen to be satisfied here). Consequently, for all n

$$e_n^{\mathrm{ran}}(S_0, F_0) \leq \|\Psi\| e_n^{\mathrm{ran}}(S, F) = e_n^{\mathrm{ran}}(S, F),$$

since $\|\Psi\| = 1$. On the other hand, it is well-known that there are constants $c_1, c_2 > 0$ such that for all n

$$c_1 n^{-\gamma-1/2} \leq e_n^{\mathrm{ran}}(S_0, F_0) \leq c_2 n^{-\gamma-1/2}$$

(see [13], 2.2.9, Prop. 2). □

Remark. Note that not only the number of information calls, but also the number of arithmetic operations of the algorithm presented above is bounded by cn.

4 Comments

Besides determining the average (mean square) error we may also estimate the distribution of the error, and in particular, the probability that the optimal rate is achieved. Due to uniform boundedness of the involved random variables we are able to show an exponential decay of the probability of the exceptional set:

Proposition 2. *There are constants $c_1, c_2 > 0$ such that for all $(f, y_0) \in F$, $n \in \mathbb{N}$ with $n \geq 2$, and for all $\tau \geq c_1$, the error of the algorithm from section 3 satisfies*

$$\mathbb{P}\left\{\|y - \bar{y}\|_\infty > \tau n^{-\gamma-1/2}\right\} \leq \exp(-c_2\tau^2). \tag{47}$$

Proof. Put

$$\nu^2 = \sum_{i=1}^{n-1} \|\eta_i\|^2_{L_\infty(\Omega,\Sigma,\mathbb{P})}. \tag{48}$$

Then for all $\theta > 0$

$$\mathbb{P}\left\{\max_{1\leq k\leq n-1}\Big|\sum_{i=1}^{k}\eta_i\Big| > \theta\right\} \leq c_1 \exp\left(-\frac{c_2\theta^2}{\nu^2}\right). \tag{49}$$

For $d = 1$ this is the Hoeffding-Azuma maximal inequality (see, e.g., [1], Remark 1 and Lemma 2, which imply (49)). The case $d > 1$ follows by considering the coordinates separately. Now (47) is a consequence of (37), (48), and (49). □

Let us mention that the proposed method is also of optimal order in the deterministic setting, in the following sense: Fix any values (realizations) of $\xi_k \in [x_{k-1}, x_k]$ $(k = 1, \dots, n-1)$. This way the algorithm becomes deterministic. Then we have

Proposition 3. *There is a constant $c > 0$ such that for all $(f, y_0) \in F$, $n \in \mathbb{N}$ with $n \geq 2$, and for all (deterministic) numbers $\xi_k \in [x_{k-1}, x_k]$*

$$\|y - \bar{y}\|_\infty \leq cn^{-\gamma}.$$

Proof. This follows readily from(37) and (40). □

Note that this way we recover the upper bound in Kacewicz's result (7) for the deterministic setting. Formally, this also fills in the cases $r = 0$, $0 \leq \varrho < 1$ not considered in [6, 7, 8]. (However, this could also be obtained by a standard analysis of the Euler method.) Concerning the lower bound, the argument used in the proof of Theorem 1 works also for the deterministic setting and the respective deterministic integration results can be found in [13], Prop. 1.3.9. Let us also mention that Kacewicz's lower bounds in [6, 7] and [10] are stronger in the sense that they hold for the smaller class of autonomous problems, which requires different proof techniques.

There is another aspect of Proposition 3: For $\gamma > 0$, the deviation from the true solution is controlled deterministically, so it suffices to assume the smoothness just in a suitable neighbourhood of the trajectories of the solutions rather than on all of $[a, b] \times \mathbb{R}^d$. (For $\gamma = 0$ one has to resort to Proposition 2.)

Comparing the rate of the deterministic (7) with that of the randomized setting (9), we see that there is always a speedup of order $n^{-1/2}$. Now consider the case $r = 0$ and $\varrho > 0$ small. Then the deterministic rate is $n^{-\varrho}$, the randomized rate is $n^{-\varrho-1/2}$, so the relative speedup $(\varrho + 1/2)/\varrho$ can be arbitrarily large for functions of low smoothness – we thus have a similar effect as for (high dimensional) integration. In the limiting case $r = \varrho = 0$ there are no deterministic algorithms with a rate convergent to zero, since $e_n^{\mathrm{det}}(S, F) \geq c > 0$ for all n, while there are randomized algorithms of convergence order $n^{-1/2}$.

With the approach presented above, other algorithms can be analyzed, as well. For example, instead of the control variate p'_k we could have taken the

one used by Kacewicz in [9] – the Taylor expansion of f, with p_k inserted. Since for the resulting algorithm a relation analogous to (26) holds (see the Lemma on p. 828 of [9]), our analysis shows that Kacewicz's original algorithm, but with just one sample per interval, is optimal, too (though the method of proof of [9] would only give the rate $n^{-\gamma}$).

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