Randomized Complexity of Vector-Valued Approximation

Stefan Heinrich
Department of Computer Science
RPTU Kaiserslautern-Landau
D-67653 Kaiserslautern, Germany

Abstract

We study the randomized n-th minimal errors (and hence the complexity) of vector valued approximation. In a recent paper by the author [Randomized complexity of parametric integration and the role of adaption I. Finite dimensional case (preprint)] a long-standing problem of Information-Based Complexity was solved: Is there a constant $c > 0$ such that for all linear problems $\mathcal{P}$ the randomized non-adaptive and adaptive n-th minimal errors can deviate at most by a factor of c? That is, does the following hold for all linear $\mathcal{P}$ and $n \in \mathbb{N}$

$$e_n^{\mathrm{ran-non}}(\mathcal{P}) \le c e_n^{\mathrm{ran}}(\mathcal{P})\,?$$

The analysis of vector-valued mean computation showed that the answer is negative. More precisely, there are instances of this problem where the gap between non-adaptive and adaptive randomized minimal errors can be (up to log factors) of the order $n^{1/8}$. This raises the question about the maximal possible deviation. In this paper we show that for certain instances of vector valued approximation the gap is $n^{1/2}$ (again, up to log factors).

1 Introduction

Let $N, N_1, N_2 \in \mathbb{N}$ and $1 \le p, q, u, v \le \infty$. We define the space L_p^N as the set of all functions $f : \mathbb{Z}[1, N] := \{1, 2, \dots, N\} \to \mathbb{K}$ with the norm

$$\|f\|_{L_p^N} = \left(\frac{1}{N}\sum_{i=1}^N |f(i)|^p\right)^{1/p} \quad (p < \infty), \quad \|f\|_{L_\infty^N} = \max_{1\le i\le N} |f(i)|.$$

and the space $L_p^{N_1}\big(L_u^{N_2}\big)$ as the set of all functions $f : \mathbb{Z}[1, N_1] \times \mathbb{Z}[1, N_2] \to \mathbb{K}$ with the norm

$$\|f\|_{L_p^{N_1}\left(L_u^{N_2}\right)} = \left\| \left(\|f_i\|_{L_u^{N_2}}\right)_{i=1}^{N_1} \right\|_{L_p^{N_1}}$$

with $f_i = (f(i, j))_{j=1}^{N_2}$ being the rows of the matrix $(f(i, j))$. In the present paper we study the complexity of approximation in the randomized setting. More precisely, we determine the order of the randomized n-th minimal errors of

$$J^{N_1,N_2} : L_p^{N_1}\big(L_u^{N_2}\big) \to L_q^{N_1}\big(L_v^{N_2}\big), \quad J^{N_1,N_2} f = f. \tag{1}$$

The input set is the unit ball of $L_p^{N_1}\big(L_u^{N_2}\big)$, the error is measured in the norm of $L_q^{N_1}\big(L_v^{N_2}\big)$ and information is standard (values of f).

It is well-known since the 80ies that for linear problems adaptive and non-adaptive n-th minimal errors can deviate at most by a factor of 2, thus for any linear problem $\mathcal{P} = (F, G, S, K, \Lambda)$ and any $n \in \mathbb{N}$

$$e_n^{\mathrm{det-non}}(S, F, G) \le 2e_n^{\mathrm{det}}(S, F, G), \tag{2}$$

see Gal and Micchelli [1], Traub and Woźniakowski [12]. The randomized analogue of this problem is as follows: Is there a constant $c > 0$ such that for all linear problems $\mathcal{P} = (F, G, S, K, \Lambda)$ and all $n \in \mathbb{N}$

$$e_n^{\mathrm{ran-non}}(S, F, G) \le ce_n^{\mathrm{ran}}(S, F, G)\,?$$

See the open problem on p. 213 of [9], and Problem 20 on p. 146 of [10]. This problem was solved recently by the author in [7], where it was shown that for some instances of vector-valued mean computation the gap between non-adaptive and adaptive randomized n-th minimal errors can be (up to log factors) of order $n^{1/8}$. This raises the question about the maximal possible deviation. In this paper we study the randomized complexity of vector valued approximation and show that for certain instances of the gap is $n^{1/2}$ (again, up to log factors), see Corollary 1.

2 Preliminaries

Throughout this paper log means $\log_2$. We denote $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbol $\mathbb{K}$ stands for the scalar field, which is either $\mathbb{R}$ or $\mathbb{C}$. We often use the same symbol $c, c_1, c_2, \dots$ for possibly different constants, even if they appear in a sequence of relations. However, some constants are supposed to have the same meaning throughout a proof – these are denoted by symbols $c(1), c(2), \dots$. The unit ball of a normed space X is denoted by B_X.

We work in the framework of IBC [8, 11], using specifically the general approach from [3, 4], see also the extended introduction in [7]. We refer to these papers for notation and background.

An abstract numerical problem $\mathcal{P}$ is given as $\mathcal{P} = (F, G, S, K, \Lambda)$, where F is a non-empty set, G a Banach space, and S is a mapping $F \to G$. The operator S is called the solution operator, it sends the input $f \in F$ of our problem to the exact solution $S(f)$. Moreover, Λ is a nonempty set of mappings from F to K, the set of information functionals, where K is any nonempty set – the set of values of information functionals.

A problem $\mathcal{P}$ is called linear, if $K = \mathbb{K}$, F is a convex and balanced subset of a linear space X over $\mathbb{K}$, S is the restriction to F of a linear operator from X to G, and each $\lambda \in \Lambda$ is the restriction to F of a linear mapping from X to $\mathbb{K}$.

In this paper we consider the linear problem

$$\mathcal{P}^{N_1,N_2} = \left(B_{L_p^{N_1}\left(L_u^{N_2}\right)}, L_q^{N_1}(L_v^{N_2}), J^{N_1,N_2}, \mathbb{K}, \Lambda\right),$$

where $\Lambda = \{\delta_{ij} : 1 \le i \le N_1,\, 1 \le j \le N_2\}$ with $\delta_{ij}(f) = f(i,j)$.

A deterministic algorithm for $\mathcal{P}$ is a tuple $A = ((L_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty)$ such that $L_1 \in \Lambda$, $\tau_0 \in \{0,1\}$, $\varphi_0 \in G$, and for $i \in \mathbb{N}$, $L_{i+1} : K^i \to \Lambda$, $\tau_i : K^i \to \{0,1\}$, and $\varphi_i : K^i \to G$ are arbitrary mappings, where K^i denotes the i-th Cartesian power of K. Given an input $f \in F$, we define $(\lambda_i)_{i=1}^\infty$ with $\lambda_i \in \Lambda$ as follows:

$$\lambda_1 = L_1, \quad \lambda_i = L_i(\lambda_1(f), \dots, \lambda_{i-1}(f)) \quad (i \ge 2).$$

Define card(A,f), the cardinality of A at input f, to be 0 if $\tau_0 = 1$. If $\tau_0 = 0$, let card(A,f) be the first integer $n \geq 1$ with $\tau_n(\lambda_1(f),\dots,\lambda_n(f)) = 1$ if there is such an n. If $\tau_0 = 0$ and no such $n \in \mathbb{N}$ exists, put card$(A,f) = +\infty$. We define the output $A(f)$ of algorithm A at input f as

$$A(f) = \begin{cases} \varphi_0 & \text{if} \quad \mathrm{card}(A,f) \in \{0,\infty\} \\ \varphi_n(\lambda_1(f),\dots,\lambda_n(f)) & \text{if} \quad 1 \leq \mathrm{card}(A,f) = n < \infty. \end{cases} \tag{3}$$

The cardinality of A is defined as $\mathrm{card}(A,F) = \sup_{f\in F} \mathrm{card}(A,f)$. Given $n \in \mathbb{N}_0$, we define $\mathscr{A}_n^{\mathrm{det}}(\mathcal{P})$ as the set of deterministic algorithms A for $\mathcal{P}$ with card$(A) \leq n$ and the deterministic n-th minimal error of S as

$$e_n^{\mathrm{det}}(S,F,G) = \inf_{A\in\mathscr{A}_n^{\mathrm{det}}(\mathcal{P})} \sup_{f\in F} \|S(f) - A(f)\|_G. \tag{4}$$

A deterministic algorithm is called non-adaptive, if all L_i and all τ_i are constant, in other words, $L_i \in \Lambda$, $\tau_i \in \{0,1\}$. The subset of non-adaptive algorithms in $\mathscr{A}_n^{\mathrm{det}}(\mathcal{P})$ is denoted by $\mathscr{A}_n^{\mathrm{det-non}}(\mathcal{P})$ and the non-adaptive deterministic n-th minimal error $e_n^{\mathrm{det-non}}(S,F,G)$ is defined in analogy with (4).

A randomized algorithm for $\mathcal{P}$ is a tuple $A = ((\Omega,\Sigma,\mathbb{P}),(A_\omega)_{\omega\in\Omega})$, where $(\Omega,\Sigma,\mathbb{P})$ is a probability space and for each $\omega \in \Omega$, A_ω is a deterministic algorithm for $\mathcal{P}$. Let $n \in \mathbb{N}_0$. Then $\mathscr{A}_n^{\mathrm{ran}}(\mathcal{P})$ stands for the class of randomized algorithms A for $\mathcal{P}$ with the following properties: For each $f \in F$ the mapping $\omega \to \mathrm{card}(A_\omega,f)$ is Σ-measurable, $\mathbb{E}\,\mathrm{card}(A_\omega,f) \leq n$, and the mapping $\omega \to A_\omega(f)$ is Σ-to-Borel measurable and $\mathbb{P}$-almost surely separably valued, i.e., there is a separable subspace G_f of G such that $\mathbb{P}\{\omega : A_\omega(f) \in G_f\} = 1$. We define the cardinality of $A \in \mathscr{A}_n^{\mathrm{ran}}(\mathcal{P})$ as $\mathrm{card}(A,F) = \sup_{f\in F} \mathbb{E}\,\mathrm{card}(A_\omega,f)$, and the randomized n-th minimal error of S as

$$e_n^{\mathrm{ran}}(S,F,G) = \inf_{A\in\mathscr{A}_n^{\mathrm{ran}}(\mathcal{P})} \sup_{f\in F} \mathbb{E}\,\|S(f) - A_\omega(f)\|_G.$$

We call a randomized algorithm $((\Omega,\Sigma,\mathbb{P}),(A_\omega)_{\omega\in\Omega})$ non-adaptive, if A_ω is non-adaptive for all $\omega \in \Omega$. Furthermore, $\mathscr{A}_n^{\mathrm{ran-non}}(\mathcal{P})$ is the subset of $\mathscr{A}_n^{\mathrm{ran}}(\mathcal{P})$ consisting of non-adaptive algorithms, and $e_n^{\mathrm{ran-non}}(S,F,G)$ denotes the non-adaptive randomized n-th minimal error.

We also need the average case setting. For the purposes of this paper we consider it only for measures which are supported by a finite subset of F. Then the underlying σ-algebra is assumed to be 2^F, therefore no measurability conditions have to be imposed on S and the involved deterministic algorithms. So let μ be a probability measure on F with finite support, let $\mathrm{card}(A,\mu) = \int_F \mathrm{card}(A,f)d\mu(f)$, and define

$$e_n^{\mathrm{avg}}(S,\mu,G) \;\; = \;\; \inf_A \int_F \|S(f) - A(f)\|_G d\mu(f),$$

where the infimum is taken over all deterministic algorithms with $\mathrm{card}(A,\mu) \leq n$. Correspondingly, $e_n^{\mathrm{avg-non}}(S,\mu,G)$ is defined. We use the following well-known results to prove lower bounds.

Lemma 1. *For every probability measure μ on F of finite support we have*

$$e_n^{\mathrm{ran}}(S,F) \geq \frac{1}{2} e_{2n}^{\mathrm{avg}}(S,\mu), \quad e_n^{\mathrm{ran-non}}(S,F) \geq \frac{1}{2} e_{2n}^{\mathrm{avg-non}}(S,\mu).$$

The types of lower bounds stated in the next lemma are well-known in IBC (see [8, 11]). For the specific form presented here we refer, e.g., to [3], Lemma 6 for statement (i), and to [6], Proposition 3.1 for (ii).

Lemma 2. *Let $\mathcal{P} = (F, G, S, K, \Lambda)$ be a linear problem, $\bar{n} \in \mathbb{N}$, and suppose there are $(f_i)_{i=1}^{\bar{n}} \subseteq F$ such that the sets $\{\lambda \in \Lambda : \lambda(f_i) \neq 0\}$ $(i = 1, \ldots, \bar{n})$ are mutually disjoint. Then the following hold for all $n \in \mathbb{N}$ with $4n < \bar{n}$:*

(i) If $\sum_{i=1}^{\bar{n}} \alpha_i f_i \in F$ for all sequences $(\alpha_i)_{i=1}^{\bar{n}} \in \{-1, 1\}^{\bar{n}}$ and μ is the distribution of $\sum_{i=1}^{\bar{n}} \varepsilon_i f_i$, where ε_i are independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = 1/2$, then

$$e_n^{\mathrm{avg}}(S, \mu) \geq \frac{1}{2} \min \left\{ \mathbb{E} \Big\| \sum_{i \in I} \varepsilon_i S f_i \Big\|_G : I \subseteq \{1, \ldots, \bar{n}\},\ |I| \geq \bar{n} - 2n \right\}.$$

(ii) If $\alpha f_i \in F$ for all $1 \leq i \leq \bar{n}$ and $\alpha \in \{-1, 1\}$, and μ is the uniform distribution on the set $\{\alpha f_i : 1 \leq i \leq \bar{n},\ \alpha \in \{-1, 1\}\}$, then

$$e_n^{\mathrm{avg}}(S, \mu) \geq \frac{1}{2} \min_{1 \leq i \leq \bar{n}} \|S f_i\|_G.$$

Finally, let θ be the mapping given by the median, that is, if $z_1^* \leq \cdots \leq z_m^*$ is the non-decreasing rearrangement of $(z_1, \ldots, z_m) \in \mathbb{R}^m$, then $\theta(z_1, \ldots, z_m)$ stands for $z^*_{(m+1)/2}$ if m is odd and $\frac{z^*_{m/2} + z^*_{m/2+1}}{2}$ if m is even. The following is well-known, see, e.g, [2].

Lemma 3. *Let $\zeta_1, \ldots, \zeta_m$ be independent, identically distributed real-valued random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$, $z \in \mathbb{R}$, $\varepsilon > 0$, and assume that $\mathbb{P}\{|z - \zeta_1| \leq \varepsilon\} \geq 3/4$. Then*

$$\mathbb{P}\{|z - \theta(\zeta_1, \ldots, \zeta_m)| \leq \varepsilon\} \geq 1 - e^{-m/8}.$$

3 An adaptive algorithm for vector valued approximation

We refer to the definition of the embedding J^{N_1,N_2} given in (1). It is easily checked by Hölder's inequality that

$$\left\| J^{N_1,N_2} \right\| = N_1^{(1/p-1/q)_+} N_2^{(1/u-1/v)_+}, \tag{5}$$

with $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$. We need the randomized norm estimation algorithm from [5]. Let $(Q, \mathcal{Q}, \varrho)$ be a probability space and let $1 \leq v < u \leq \infty$. For $n \in \mathbb{N}$ define $A_n^1 = (A^1_{n,\omega})_{\omega \in \Omega}$ by setting for $\omega \in \Omega$ and $f \in L_u(Q, \mathcal{Q}, \varrho)$

$$A^1_{n,\omega}(f) \quad = \quad \left(\frac{1}{n} \sum_{i=1}^{n} |f(\xi_i(\omega_2))|^v \right)^{1/v}, \tag{6}$$

where ξ_i are independent Q-valued random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$ with distribution ϱ. The following is essentially Proposition 6.3 of [5], for a self-contained proof we refer to [7].

Proposition 1. *Let $1 \leq v < u \leq \infty$. Then there is a constant $c > 0$ such that for all probability spaces $(Q, \mathcal{Q}, \varrho)$, $f \in L_u(Q, \mathcal{Q}, \varrho)$, and $n \in \mathbb{N}$*

$$\mathbb{E} \left| \|f\|_{L_v(Q,\mathcal{Q},\varrho)} - A^1_{n,\omega}(f) \right| \quad \leq \quad c n^{\max(1/u-1/v,-1/2)} \|f\|_{L_u(Q,\mathcal{Q},\varrho)}. \tag{7}$$

The algorithm for approximation of J^{N_1,N_2} will only be defined for the case that $1 \le p < q \le \infty$ and $1 \le v < u \le \infty$ (it turns out that for the other cases the zero algorithm is of optimal order). Define for $m, n \in \mathbb{N}$, $n < N_1 N_2$ an adaptive algorithm $A^2_{n,m,\omega}$. Let $f \in L_p^{N_1}(L_u^{N_2})$ and set $f_i = (f(i,j))_{j=1}^{N_2}$. Let $\left\{\xi_{jk} : 1 \le j \le \left\lceil \frac{n}{N_1} \right\rceil, 1 \le k \le m\right\}$ be independent random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$ uniformly distributed over $\{1, \dots, N_2\}$.

We apply algorithm $A^1_{n,\omega}$, see (6), to estimate $\|f_i\|_{L_v^{N_2}}$ by setting for $\omega \in \Omega$, $1 \le i \le N_1$, $1 \le k \le m$

$$a_{ik}(\omega) = \left(\left\lceil \frac{n}{N_1} \right\rceil^{-1} \sum_{1 \le j \le \left\lceil \frac{n}{N_1} \right\rceil} |f_i(\xi_{jk}(\omega))|^v \right)^{1/v}, \quad \tilde{a}_i(\omega) = \theta\big((a_{ik}(\omega))_{k=1}^m\big).$$

Let $\tilde{a}_{\pi(1)} \ge \dots \ge \tilde{a}_{\pi(N_1)}$ be a non-increasing rearrangement of $(\tilde{a}_i)$, with π a permutation. Then the output $A^2_{n,m,\omega}(f) \in L_q^{N_1}(L_v^{N_2})$ of the algorithm is defined as

$$A^2_{n,m,\omega}(f) = b = (b_i(\omega))_{i=1}^{N_1}, \quad b_{\pi(i)}(\omega) = \begin{cases} f_{\pi(i)} & \text{if} \quad i \le \left\lceil \frac{n}{N_2} \right\rceil \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

(note that the assumption on n implies $\left\lceil \frac{n}{N_2} \right\rceil \le N_1$). If $\frac{1}{v} - \frac{1}{u} > \frac{1}{2}$, we use an iterated version $(A^3_{n,m,\omega})_{\omega \in \Omega}$, where $(\Omega, \Sigma, \mathbb{P}) = (\Omega_1, \Sigma_1, \mathbb{P}_1) \times (\Omega_2, \Sigma_2, \mathbb{P}_2)$ with $(\Omega_\iota, \Sigma_\iota, \mathbb{P}_\iota)$ $(\iota = 1, 2)$ being probability spaces. We define

$$A^3_{n,m,\omega}(f) = A^2_{n,m,\omega_1}(f) + A^2_{n,m,\omega_2}(f - A^2_{n,m,\omega_1}(f)) \quad (\omega = (\omega_1, \omega_2)). \tag{9}$$

The constants in the subsequent statements and proofs are independent of the parameters n, N_1,N_2, and m. This is also made clear by the order of quantifiers in the respective statements.

Proposition 2. *Let $1 \le p < q \le \infty$, $1 \le v < u \le \infty$, and $1 \le w < \infty$. Then there exist constants $c_1 > 1$, $c_2 > 0$ such that the following hold for all $m, n, N_1, N_2 \in \mathbb{N}$ with $n < N_1 N_2$ and $f \in L_p^{N_1}(L_u^{N_2})$:*

$$\operatorname{card}(A^2_{n,m,\omega}) \le (m+1)n + mN_1 + N_2, \quad \operatorname{card}(A^3_{n,m,\omega}) = 2\operatorname{card}(A^2_{n,m,\omega}). \tag{10}$$

Furthermore, setting $A_{n,m,\omega} = A^2_{n,m,\omega}$ if $\frac{1}{v} - \frac{1}{u} \le \frac{1}{2}$ and $A_{n,m,\omega} = A^3_{n,m,\omega}$ if $\frac{1}{v} - \frac{1}{u} > \frac{1}{2}$, we have for $m \ge c_1 \log(N_1 + N_2)$

$$\begin{aligned} &\left(\mathbb{E}\, \|f - A_{n,m,\omega}(f)\|^w_{L_q^{N_1}(L_v^{N_2})}\right)^{1/w} \\ &\le c_2 N_1^{1/p-1/q} \left(\left\lceil \frac{n}{N_1} \right\rceil^{1/u-1/v} + \left\lceil \frac{n}{N_2} \right\rceil^{1/q-1/p} \right) \|f\|_{L_p^{N_1}(L_u^{N_2})}. \end{aligned} \tag{11}$$

Proof. The total number of samples in $A^2_{n,m,\omega}$ is

$$mN_1 \left\lceil \frac{n}{N_1} \right\rceil + N_2 \left\lceil \frac{n}{N_2} \right\rceil \le mn + mN_1 + n + N_2,$$

which gives (10). For $n < \max(N_1, N_2)$ relation (11) follows from (5). Hence in the sequel we assume $n \ge \max(N_1, N_2)$. Fix $f \in L_p^{N_1}(L_u^{N_2})$. First we consider the case $\frac{1}{v} - \frac{1}{u} \le \frac{1}{2}$. By Proposition 1, where here the respective constant is denoted by $c(0)$,

$$\mathbb{E}\left| \|f_i\|_{L_v^{N_2}} - a_{ik} \right| \le c(0) \left(\frac{n}{N_1} \right)^{1/u-1/v} \|f_i\|_{L_u^{N_2}}, \tag{12}$$

and therefore,

$$\mathbb{P}\left\{\omega\in\Omega:\ \Big|\|f_i\|_{L_v^{N_2}}-a_{ik}(\omega)\Big|\le 4c(0)\left(\frac{n}{N_1}\right)^{1/u-1/v}\|f_i\|_{L_u^{N_2}}\right\}\ge\frac{3}{4}. \tag{13}$$

Let $c(1)=\frac{8(w+1)}{\log e}>1$ (recall that log always means $\log_2$), then $m\ge c(1)\log(N_1+N_2)$ implies $e^{-m/8}\le(N_1+N_2)^{-w-1}$. From (13) and Lemma 3 we conclude

$$\mathbb{P}\left\{\omega\in\Omega:\ \Big|\|f_i\|_{L_v^{N_2}}-\tilde a_i(\omega)\Big|\le 4c(0)\left(\frac{n}{N_1}\right)^{1/u-1/v}\|f_i\|_{L_u^{N_2}}\right\}\ge 1-(N_1+N_2)^{-w-1}.$$

Let

$$\Omega_0 \;=\; \left\{\omega\in\Omega:\Big|\|f_i\|_{L_v^{N_2}}-\tilde a_i(\omega)\Big|\le 4c(0)\left(\frac{n}{N_1}\right)^{1/u-1/v}\|f_i\|_{L_u^{N_2}}\ (1\le i\le N_1)\right\}, \tag{14}$$

thus

$$\mathbb{P}(\Omega_0)\ge 1-(N_1+N_2)^{-w}. \tag{15}$$

Fix $\omega\in\Omega_0$. Then by (14) for all i

$$\tilde a_i(\omega)\le c\|f_i\|_{L_u^{N_2}}. \tag{16}$$

Consequently,

$$\left(\frac{1}{N_1}\sum_{i=1}^{N_1}\tilde a_i(\omega)^p\right)^{1/p}\le c\left(\frac{1}{N_1}\sum_{i=1}^{N_1}\|f_i\|^p_{L_u^{N_2}}\right)^{1/p}=c\|f\|_{L_p^{N_1}(L_u^{N_2})}. \tag{17}$$

Let $M=\left\lceil\frac{n}{N_2}\right\rceil$. It follows that

$$c\|f\|_{L_p^{N_1}(L_u^{N_2})}\ge\left(\frac{1}{N_1}\sum_{i=1}^{M}\tilde a^p_{\pi(i)}\right)^{1/p}\ge\left(\frac{M}{N_1}\right)^{1/p}\tilde a_{\pi(M)}, \tag{18}$$

thus for $i>M$

$$\tilde a_{\pi(i)}\le\tilde a_{\pi(M)}\le c\left(\frac{N_1}{M}\right)^{1/p}\|f\|_{L_p^{N_1}(L_u^{N_2})}\le c\left(\frac{N_1N_2}{n}\right)^{1/p}\|f\|_{L_p^{N_1}(L_u^{N_2})}. \tag{19}$$

Furthermore, by (8), for $i\le M$

$$\big\|f_{\pi(i)}-b_{\pi(i)}(\omega)\big\|_{L_v^{N_2}} \;=\; 0. \tag{20}$$

Let

$$I(\omega):=\left\{1\le i\le N_1:\ \tilde a_{\pi(i)}(\omega)\le\frac{\|f_{\pi(i)}\|_{L_v^{N_2}}}{2}\right\}, \tag{21}$$

hence we conclude from (8) and (14) for $i>M$, $i\in I(\omega)$

$$\begin{aligned}\big\|f_{\pi(i)}-b_{\pi(i)}(\omega)\big\|_{L_v^{N_2}} &= \|f_{\pi(i)}\|_{L_v^{N_2}}\le 2\big(\|f_{\pi(i)}\|_{L_v^{N_2}}-\tilde a_{\pi(i)}(\omega)\big)\\ &\le c\left(\frac{n}{N_1}\right)^{1/u-1/v}\|f_{\pi(i)}\|_{L_u^{N_2}}.\end{aligned} \tag{22}$$

On the other hand, we have by (19) for $i > M$, $i \notin I(\omega)$

$$\begin{aligned}\left\|f_{\pi(i)} - b_{\pi(i)}(\omega)\right\|_{L_v^{N_2}} &= \left\|f_{\pi(i)}\right\|_{L_v^{N_2}} < 2\tilde{a}_{\pi(i)}(\omega) = 2\tilde{a}_{\pi(i)}(\omega)^{p/q}\tilde{a}_{\pi(i)}(\omega)^{1-p/q} \\ &\le c\tilde{a}_{\pi(i)}(\omega)^{p/q}\left(\frac{N_1N_2}{n}\right)^{1/p-1/q}\|f\|^{1-p/q}_{L_p^{N_1}(L_u^{N_2})} \qquad (23)\end{aligned}$$

(with the convention $0^0 = 1$). Combining (20), (22), and (23), we get for $1 \le i \le N_1$

$$\left\|f_i - b_i(\omega)\right\|_{L_v^{N_2}} \le c\left(\frac{n}{N_1}\right)^{1/u-1/v}\|f_i\|_{L_u^{N_2}} + c\tilde{a}_i(\omega)^{p/q}\left(\frac{N_1N_2}{n}\right)^{1/p-1/q}\|f\|^{1-p/q}_{L_p^{N_1}(L_u^{N_2})}.$$

Together with (17) we obtain for $\omega \in \Omega_0$,

$$\begin{aligned}&\left\|f - (b_i(\omega))_{i=1}^{N_1}\right\|_{L_q^{N_1}(L_v^{N_2})} \\ &\le c\left(\frac{n}{N_1}\right)^{1/u-1/v}\left\|\left(\|f_i\|_{L_u^{N_2}}\right)_{i=1}^{N_1}\right\|_{L_q^{N_1}} + c\left(\frac{N_1N_2}{n}\right)^{1/p-1/q}\left\|\left(\tilde{a}_i(\omega)^{p/q}\right)_{i=1}^{N_1}\right\|_{L_q^{N_1}}\|f\|^{1-p/q}_{L_p^{N_1}(L_u^{N_2})} \\ &\le c\left(\frac{n}{N_1}\right)^{1/u-1/v}N_1^{1/p-1/q}\left\|\left(\|f_i\|_{L_u^{N_2}}\right)_{i=1}^{N_1}\right\|_{L_p^{N_1}} \\ &\quad +c\left(\frac{N_1N_2}{n}\right)^{1/p-1/q}\left(\frac{1}{N_1}\sum_{i=1}^{N_1}\tilde{a}_i(\omega)^p\right)^{1/q}\|f\|^{1-p/q}_{L_p^{N_1}(L_u^{N_2})} \\ &\le cN_1^{1/p-1/q}\left(\left(\frac{n}{N_1}\right)^{1/u-1/v} + \left(\frac{n}{N_2}\right)^{1/q-1/p}\right)\|f\|_{L_p^{N_1}(L_u^{N_2})}. \qquad (24)\end{aligned}$$

To estimate the error on $\Omega \setminus \Omega_0$ we note that by (8) for all $\omega \in \Omega$, b_i is either f_i or zero. Consequently

$$\left\|f - b(\omega)\right\|_{L_q^{N_1}(L_v^{N_2})} \le \left\|f\right\|_{L_q^{N_1}(L_v^{N_2})} \le N_1^{1/p-1/q}\|f\|_{L_p^{N_1}(L_u^{N_2})}, \qquad (25)$$

and therefore, using (15),

$$\begin{aligned}&\left(\int_{\Omega\setminus\Omega_0}\left\|f - (b_i(\omega))_{i=1}^{N_1}\right\|^w_{L_q^{N_1}(L_v^{N_2})}d\mathbb{P}(\omega)\right)^{1/w} \\ &\le N_1^{1/p-1/q}(N_1+N_2)^{-1}\|f\|_{L_p^{N_1}(L_u^{N_2})} \le N_1^{1/p-1/q}\left(\frac{n}{N_1}\right)^{1/q-1/p}\|f\|_{L_p^{N_1}(L_u^{N_2})},\end{aligned}$$

the last relation being a consquence of $n < N_1N_2$. Together with (24) this shows (11) under the assumption $\frac{1}{v} - \frac{1}{u} \le \frac{1}{2}$.

Finally we consider the case $\frac{1}{v} - \frac{1}{u} > \frac{1}{2}$. We define q_1, v_1 by

$$\frac{1}{q_1} = \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right), \quad \frac{1}{v_1} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{v}\right), \qquad (26)$$

then $1 \le p < q_1 < q$, $v < v_1 < u$, $\frac{1}{v_1} - \frac{1}{u} \le \frac{1}{2}$, and $\frac{1}{v} - \frac{1}{v_1} \le \frac{1}{2}$, so we conclude from the already shown case of (11)

$$\begin{aligned}&\left(\mathbb{E}_{\omega_1}\|f - A^2_{n,m,\omega_1}(f)\|^w_{L_{q_1}^{N_1}(L_{v_1}^{N_2})}\right)^{1/w} \\ &\le cN_1^{1/p-1/q_1}\left(\left(\frac{n}{N_1}\right)^{1/u-1/v_1} + \left(\frac{n}{N_2}\right)^{1/q_1-1/p}\right)\|f\|_{L_p^{N_1}(L_u^{N_2})} \qquad (27)\end{aligned}$$

and, with $g = f - A^2_{n,m,\omega_1}(f)$,

$$\begin{aligned}
&\left(\mathbb{E}_{\omega_2}\|g - A^2_{n,m,\omega_2}(g)\|^w_{L_q^{N_1}(L_v^{N_2})}\right)^{1/w} \\
&\quad \le \; cN_1^{1/q_1-1/q}\left(\left(\frac{n}{N_1}\right)^{1/v_1-1/v} + \left(\frac{n}{N_2}\right)^{1/q-1/q_1}\right)\|g\|_{L_{q_1}^{N_1}(L_{v_1}^{N_2})}. \qquad (28)
\end{aligned}$$

From (9), (27), and (28) we obtain

$$\begin{aligned}
&\left(\mathbb{E}_{\omega_1}\mathbb{E}_{\omega_2}\|f - A^3_{n,m,(\omega_1,\omega_2)}(f)\|^w_{L_q^{N_1}(L_v^{N_2})}\right)^{1/w} \\
&= \left(\mathbb{E}_{\omega_1}\mathbb{E}_{\omega_2}\|f - A^2_{n,m,\omega_1}(f) - A^2_{n,m,\omega_2}(f - A^2_{n,m,\omega_1}(f))\|^w_{L_q^{N_1}(L_v^{N_2})}\right)^{1/w} \\
&\le cN_1^{1/q_1-1/q}\left(\left(\frac{n}{N_1}\right)^{1/v_1-1/v} + \left(\frac{n}{N_2}\right)^{1/q-1/q_1}\right)\left(\mathbb{E}_{\omega_1}\|f - A^2_{n,m,\omega_1}(f)\|^w_{L_{q_1}^{N_1}(L_{v_1}^{N_2})}\right)^{1/w} \\
&\le cN_1^{1/q_1-1/q}\left(\left(\frac{n}{N_1}\right)^{1/v_1-1/v} + \left(\frac{n}{N_2}\right)^{1/q-1/q_1}\right) \\
&\quad \times N_1^{1/p-1/q_1}\left(\left(\frac{n}{N_1}\right)^{1/u-1/v_1} + \left(\frac{n}{N_2}\right)^{1/q_1-1/p}\right)\|f\|_{L_p^{N_1}(L_u^{N_2})} \\
&= cN_1^{1/p-1/q}\left(\left(\frac{n}{N_1}\right)^{\frac{1}{2}(1/u-1/v)} + \left(\frac{n}{N_2}\right)^{\frac{1}{2}(1/q-1/p)}\right)^2\|f\|_{L_p^{N_1}(L_u^{N_2})} \\
&\le cN_1^{1/p-1/q}\left(\left(\frac{n}{N_1}\right)^{1/u-1/v} + \left(\frac{n}{N_2}\right)^{1/q-1/p}\right)\|f\|_{L_p^{N_1}(L_u^{N_2})}. \qquad (29)
\end{aligned}$$

This gives (11) and concludes the proof. □

4 Lower bounds and complexity

Proposition 3. *Let $1 \le p, q, u, v \le \infty$. Then there exist constants $0 < c_0 < 1$, $c_1 \dots c_6 > 0$ such that for all $n, N_1, N_2 \in \mathbb{N}$, with $n < c_0 N_1 N_2$ there exist probability measures $\mu^{(i)}_{n,N_1,N_2}$ $(1 \le i \le 6)$ with finite support in $B_{L_p^{N_1}(L_u^{N_2})}$ such that*

$$\begin{aligned}
e_n^{\rm avg}(J^{N_1,N_2}, \mu^{(1)}_{n,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) &\ge c_1 N_1^{1/p-1/q}\left\lceil\frac{n}{N_1}\right\rceil^{1/u-1/v} && (30)\\
e_n^{\rm avg}(J^{N_1,N_2}, \mu^{(2)}_{n,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) &\ge c_2 N_1^{1/p-1/q}\left\lceil\frac{n}{N_2}\right\rceil^{1/q-1/p} && (31)\\
e_n^{\rm avg}(J^{N_1,N_2}, \mu^{(3)}_{n,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) &\ge c_4 N_1^{1/p-1/q} N_2^{1/u-1/v} && (32)\\
e_n^{\rm avg}(J^{N_1,N_2}, \mu^{(4)}_{n,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) &\ge c_3 && (33)\\
e_n^{\rm avg}(J^{N_1,N_2}, \mu^{(5)}_{n,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) &\ge c_5\left\lceil\frac{n}{N_1}\right\rceil^{1/u-1/v} && (34)\\
e_n^{\rm avg-non}(J^{N_1,N_2}, \mu^{(6)}_{n,N_1,N_2}), L_q^{N_1}(L_v^{N_2})) &\ge c_6 N_1^{1/p-1/q}. && (35)
\end{aligned}$$

Proof. We set $c_0 = \frac{1}{21}$ and let $n \in \mathbb{N}$ be such that

$$1 \le n < \frac{N_1 N_2}{21}. \tag{36}$$

Define for L with $1 \le L \le N_2$ disjoint subsets of $\{1, \dots, N_2\}$ by setting

$$D_j = \left\{ (j-1) \left\lfloor \frac{N_2}{L} \right\rfloor + 1, \dots, j \left\lfloor \frac{N_2}{L} \right\rfloor \right\}, \quad (j = 1, \dots, L), \tag{37}$$

then

$$\frac{N_2}{2L} < \left\lfloor \frac{N_2}{L} \right\rfloor = |D_j| \le \frac{N_2}{L}. \tag{38}$$

To show (30), we put

$$L = \left\lfloor \frac{4n}{N_1} \right\rfloor + 1, \tag{39}$$

thus

$$\left\lceil \frac{4n}{N_1} \right\rceil < L \le 5 \left\lceil \frac{n}{N_1} \right\rceil. \tag{40}$$

By (36), $\frac{4n}{N_1} < N_2$, which together with (39) gives $L \le N_2$, as required above. We conclude from (38) and (40)

$$\frac{N_2}{10} \left\lceil \frac{n}{N_1} \right\rceil^{-1} < |D_j| \le N_2 \left\lceil \frac{n}{N_1} \right\rceil^{-1}. \tag{41}$$

Let for $1 \le i \le N_1$ and $1 \le j \le L$

$$\psi_{ij}(s,t) = \begin{cases} N_1^{1/p} N_2^{1/u} |D_j|^{-1/u} & \text{if } s = i \quad \text{and} \quad t \in D_j, \\ 0 & \text{otherwise,} \end{cases} \tag{42}$$

and let $\mu^{(1)}_{n,N_1,N_2}$ be the unifrom distribution on the set

$$\{\alpha \psi_{ij} : \, i = 1, \dots, N_1, \, j = 1, \dots, L, \, \alpha = \pm 1\} \subset B_{L_p^{N_1}(L_u^{N_2})}.$$

Recall that by (40), $LN_1 > 4n$, so from Lemma 2(ii) and relation (41) we conclude

$$\begin{aligned} e_n^{\mathrm{avg}}(J^{N_1,N_2}, \mu^{(1)}_{n,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) &\ge \frac{1}{2} \left\| J^{N_1,N_2} \psi_{1,1} \right\|_{L_q^{N_1}(L_v^{N_2})} = \frac{1}{2} N_1^{1/p-1/q} N_2^{1/u-1/v} |D_1|^{1/v-1/u} \\ &\ge c N_1^{1/p-1/q} \left\lceil \frac{n}{N_1} \right\rceil^{1/u-1/v}, \end{aligned}$$

thus (30).

To prove (31), we set

$$M = \left\lfloor \frac{4n}{N_2} \right\rfloor + 1, \tag{43}$$

so similarly to the above,

$$\left\lceil \frac{4n}{N_2} \right\rceil < M \le 5 \left\lceil \frac{n}{N_2} \right\rceil \tag{44}$$

and $M \leq N_1$. Now define for $1 \leq i \leq M$ and $1 \leq j \leq N_2$

$$\psi_{ij}(s,t) = \begin{cases} N_1^{1/p} M^{-1/p}, & \text{if } s = i \quad \text{and} \quad t = j \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$

Let $(\varepsilon_{ij})_{i=1,j=1}^{M,N_2}$ be independent symmetric Bernoulli random variables and let $\mu^{(2)}_{n,N_1,N_2}$ be the distribution of $\sum_{i=1}^{M} \sum_{j=1}^{N_2} \varepsilon_{ij}\psi_{ij}$. Then $\mu^{(2)}_{n,N_1,N_2}$ is concentrated on $B_{L_p^{N_1}(L_u^{N_2})}$. Since by (44), $MN_2 > 4n$, we can apply Lemma 2. So let $\mathcal{K}$ be any subset of $\{(i,j) \,:\, 1 \leq i \leq M,\, 1 \leq j \leq N_2\}$ with $|\mathcal{K}| \geq MN_2 - 2n$. Then

$$|\mathcal{K}| \geq \frac{1}{2} MN_2. \tag{46}$$

For $1 \leq i \leq M$ let

$$\mathcal{K}_i = \{1 \leq j \leq N_2 \,:\, (i,j) \in \mathcal{K}\}, \quad I := \left\{1 \leq i \leq M \,:\, |\mathcal{K}_i| \geq \frac{N_2}{4}\right\}. \tag{47}$$

Then $|I| \geq \frac{M}{4}$ and we get from (44) and (47)

$$\begin{aligned}
\mathbb{E}\Big\| \sum_{(i,j)\in\mathcal{K}} \varepsilon_{ij} J^{N_1,N_2} \psi_{ij} \Big\|_{L_q^{N_1}(L_v^{N_2})} &\geq \mathbb{E}\Big\| \sum_{i\in I} \sum_{j\in\mathcal{K}_i} \varepsilon_{ij}\psi_{ij} \Big\|_{L_q^{N_1}(L_v^{N_2})} \geq 4^{-1/v} N_1^{1/p} M^{-1/p} |I|^{1/q} N_1^{-1/q} \\
&\geq c N_1^{1/p-1/q} M^{1/q-1/p} \geq c N_1^{1/p-1/q} \left\lceil \frac{n}{N_2} \right\rceil^{1/q-1/p}
\end{aligned}$$

and from Lemma 2 (i)

$$\begin{aligned}
&e_n^{\mathrm{avg}}(J^{N_1,N_2}, \mu^{(2)}_{n,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) \\
&\geq \frac{1}{2} \min_{|\mathcal{K}| \geq MN_2 - 2n} \mathbb{E}\Big\| \sum_{(i,j)\in\mathcal{K}} \varepsilon_{ij} J^{N_1,N_2}\psi_{ij} \Big\|_{L_q^{N_1}(L_v^{N_2})} \geq c N_1^{1/p-1/q} \left\lceil \frac{n}{N_2} \right\rceil^{1/q-1/p},
\end{aligned}$$

thus (31).

We derive relation (32) directly from (30) and (33) from (31). Setting $n_1 = \lceil c_0 N_1 N_2 \rceil - 1$ and recalling from (36) that $1 < c_0 N_1 N_2$, we get $\frac{c_0}{2} N_1 N_2 \leq n_1 < c_0 N_1 N_2$. Consequently,

$$\frac{c_0}{2} N_1 \leq \left\lceil \frac{n_1}{N_2} \right\rceil < (c_0+1) N_1, \quad \frac{c_0}{2} N_2 \leq \left\lceil \frac{n_1}{N_1} \right\rceil < (c_0+1) N_2. \tag{48}$$

We set

$$\mu^{(3)}_{n,N_1,N_2} = \mu^{(1)}_{n_1,N_1,N_2}, \quad \mu^{(4)}_{n,N_1,N_2} = \mu^{(2)}_{n_1,N_1,N_2}.$$

Furthermore, since $n < c_0 N_1 N_2$ we have $n \leq n_1$, hence by monotonicity, (30), and (48)

$$\begin{aligned}
&e_n^{\mathrm{avg}}(J^{N_1,N_2}, \mu^{(3)}_{n,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) = e_n^{\mathrm{avg}}(J^{N_1,N_2}, \mu^{(1)}_{n_1,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) \\
&\geq e_{n_1}^{\mathrm{avg}}(J^{N_1,N_2}, \mu^{(1)}_{n_1,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) \geq c N_1^{1/p-1/q} \left\lceil \frac{n_1}{N_1} \right\rceil^{1/u-1/v} \geq c N_1^{1/p-1/q} N_2^{1/u-1/v},
\end{aligned}$$

thus (32). Similarly, from (31)

$$\begin{aligned} e_n^{\rm avg}(J^{N_1,N_2},\mu^{(4)}_{n,N_1,N_2},L_q^{N_1}(L_v^{N_2})) &= e_n^{\rm avg}(J^{N_1,N_2},\mu^{(2)}_{n_1,N_1,N_2},L_q^{N_1}(L_v^{N_2})) \\ \ge \ e_{n_1}^{\rm avg}(J^{N_1,N_2},\mu^{(2)}_{n_1,N_1,N_2},L_q^{N_1}(L_v^{N_2})) &\ge cN_1^{1/p-1/q}\left\lceil\frac{n_1}{N_2}\right\rceil^{1/q-1/p} \ge c, \end{aligned}$$

which is (33).

For the proof of inequalities (34) and (35) we can assume $n\ge N_1$, because for $n<N_1$ the already shown relation (33) implies (34), while (30) gives (35). We set

$$L = 4\left\lceil\frac{4n}{N_1}\right\rceil + 1, \tag{49}$$

hence by (36)

$$L \le \frac{16n}{N_1} + 5 \le \frac{21n}{N_1} \le N_2.$$

To prove (34), we use again the blocks D_j $(j=1,\dots,L)$ given by (37) and define $\psi_j\in B_{L_u^{N_2}}$ by

$$\psi_j(t) = \begin{cases} N_2^{1/u}|D_j|^{-1/u} & \text{if} \quad t\in D_j, \\ 0 & \text{otherwise.}\end{cases} \tag{50}$$

Let μ_1 be the counting measure on $\{\pm\psi_j : 1\le j\le L\}\subset L_u^{N_2}$. Then we set $\mu^{(5)}_{n,N_1,N_2}=\mu_1^{N_1}$, the N_1-th power of μ_1, considered as a measure on $L_p^{N_1}(L_u^{N_2})$. This measure has its support in $B_{L_p^{N_1}(L_u^{N_2})}$, and with $J^{N_2}: L_u^{N_2}\to L_v^{N_2}$ being the identical embedding, Corollary 2.4 of [7] gives

$$e_n^{\rm avg}(J^{N_1,N_2},\mu^{(5)}_{n,N_1,N_2},L_q^{N_1}(L_v^{N_2})) \ge 2^{-1-1/q} e^{\rm avg}_{\lceil\frac{4n}{N_1}\rceil}(J^{N_2},\mu_1,L_v^{N_2}). \tag{51}$$

By Lemma 2(ii) with $\bar n = L$, (38), and (49)

$$e^{\rm avg}_{\lceil\frac{4n}{N_1}\rceil}(J^{N_2},\mu_1,L_v^{N_2}) \ge \frac12\|J^{N_2}\psi_1\|_{L_v^{N_2}} = \frac12 N_2^{1/u-1/v}|D_j|^{1/v-1/u} \ge c\left\lceil\frac{n}{N_1}\right\rceil^{1/u-1/v},$$

which together with (51) gives (34).

Finally we turn to (35), where we set

$$\psi_j = N_1^{1/p}\chi_{D_j}\in L_u^{N_2} \quad (j=1,\dots,L), \tag{52}$$

with D_j given by (37) and L by (49). Let $(\varepsilon_j)_{j=1}^L$ be independent symmetric Bernoulli random variables and let μ_1 be the distribution of $\sum_{j=1}^L\varepsilon_j\psi_j$. We define a measure $\mu^{(6)}_{n,N_1,N_2}$ on $B_{L_p^{N_1}(L_u^{N_2})}$ as follows: Let $\Phi_k: L_u^{N_2}\to L_p^{N_1}(L_u^{N_2})$ be the identical embedding onto the k-th component of the space $L_p^{N_1}(L_u^{N_2})$, that is, for $g\in L_u^{N_2}$, $\Phi_k(g)=f$, with $f(k,j)=g(j)$ and $f(i,j)=0$ for $i\ne k$. We define the measure $\mu^{(6)}_{n,N_1,N_2}$ on $L_p^{N_1}(L_u^{N_2})$ by setting for a set $C\subset L_p^{N_1}(L_u^{N_2})$

$$\mu^{(6)}_{n,N_1,N_2}(C) = N_1^{-1}\sum_{i=1}^{N_1}\mu_1(\Phi_i^{-1}(C)), \tag{53}$$

thus by (52), $\mu^{(6)}_{n,N_1,N_2}$ is of finite support in $B_{L_p^{N_1}(L_u^{N_2})}$. Now Corollary 2.6 of [7] yields

$$e_n^{\mathrm{avg-non}}(J^{N_1,N_2},\mu^{(6)}_{n,N_1,N_2},L_q^{N_1}(L_v^{N_2})) \ge \frac{1}{2}N_1^{-1/q}e^{\mathrm{avg-non}}_{\lceil \frac{2n}{N_1}\rceil}(J^{N_2},\mu_1,L_v^{N_2}). \tag{54}$$

By Lemma 2(i), (38), (49), and (52)

$$\begin{aligned} e^{\mathrm{avg-non}}_{\lceil \frac{2n}{N_1}\rceil}(J^{N_2},\mu_1,L_v^{N_2}) &\ge e^{\mathrm{avg}}_{\lceil \frac{2n}{N_1}\rceil}(J^{N_2},\mu_1,L_v^{N_2}) \\ \ge\ & \frac{1}{2}\min\left\{\mathbb{E}\left\|\sum_{i\in I}\varepsilon_i J^{N_2}\psi_i\right\|_{L_v^{N_2}} : I\subseteq\{1,\dots,L\},\ |I|\ge L-2\left\lceil\frac{2n}{N_1}\right\rceil\right\} \ge cN_1^{1/p}. \end{aligned}$$

Inserting this into (54) finally yields (35). □

Theorem 1. *Let $1\le p,q,u,v\le\infty$. Then there exists constants $0<c_0<1$, $c_1,\dots,c_6>0$, such that for all $n,N_1,N_2\in\mathbb{N}$ with $n<c_0N_1N_2$ the following hold: If $p\ge q$ or $u\le v$, then*

$$\begin{aligned} c_1N_1^{(1/p-1/q)_+}N_2^{(1/u-1/v)_+} &\le e_n^{\mathrm{ran}}\left(J^{N_1,N_2},B_{L_p^{N_1}(L_u^{N_2})},L_q^{N_1}(L_v^{N_2})\right) \\ \le\ & e_n^{\mathrm{ran-non}}\left(J^{N_1,N_2},B_{L_p^{N_1}(L_u^{N_2})},L_q^{N_1}(L_v^{N_2})\right) \le c_2N_1^{(1/p-1/q)_+}N_2^{(1/u-1/v)_+}. \end{aligned} \tag{55}$$

If $p<q$ and $u>v$, then

$$\begin{aligned} c_3N_1^{1/p-1/q}\left(\left\lceil\frac{n}{N_1}\right\rceil^{1/u-1/v}+\left\lceil\frac{n}{N_2}\right\rceil^{1/q-1/p}\right) &\le e_n^{\mathrm{ran}}\left(J^{N_1,N_2},B_{L_p^{N_1}(L_u^{N_2})},L_q^{N_1}(L_v^{N_2})\right) \\ \le\ c_4N_1^{1/p-1/q}&\left(\left\lceil\frac{n}{N_1\log(N_1+N_2)}\right\rceil^{1/u-1/v}+\left\lceil\frac{n}{N_2\log(N_1+N_2)}\right\rceil^{1/q-1/p}\right) \end{aligned} \tag{56}$$

and

$$c_5N_1^{1/p-1/q} \le e_n^{\mathrm{ran-non}}\left(J^{N_1,N_2},B_{L_p^{N_1}(L_u^{N_2})},L_q^{N_1}(L_v^{N_2})\right) \le c_6N_1^{1/p-1/q}. \tag{57}$$

Proof. The upper bounds in (55) and (57) are a consequence of (5), just using the zero algorithm. If $n<6(N_1+N_2)\lceil c(1)\log(N_1+N_2)\rceil$, where $c(1)>1$ is the constant c_1 from Proposition 2, the upper bound of (56) follows from (5), as well. Now assume

$$n\ge 6(N_1+N_2)\lceil c(1)\log(N_1+N_2)\rceil \tag{58}$$

and set

$$m=\lceil c(1)\log(N_1+N_2)\rceil,\quad \tilde n=\left\lfloor\frac{n}{6m}\right\rfloor. \tag{59}$$

We use Proposition 2 with $\tilde n$ instead of n, so by (10) and (59)

$$\mathrm{card}(A^2_{\tilde n,\omega})\le(m+1)\tilde n+mN_1+N_2\le 2m\tilde n+m(N_1+N_2)\le 3m\tilde n\le\frac{n}{2}, \tag{60}$$

consequently, $\mathrm{card}(A^3_{\tilde n,\omega})\le 2\,\mathrm{card}(A^2_{\tilde n,\omega})\le n$ and therefore

$$e_n^{\mathrm{ran}}\left(J^{N_1,N_2},B_{L_p^{N_1}(L_u^{N_2})},L_q^{N_1}(L_v^{N_2})\right) \le cN_1^{1/p-1/q}\left(\left\lceil\frac{\tilde n}{N_1}\right\rceil^{1/u-1/v}+\left\lceil\frac{\tilde n}{N_2}\right\rceil^{1/q-1/p}\right). \tag{61}$$

Furthermore, we obtain from (58) and (59)

$$\left\lceil \frac{\tilde{n}}{N_i} \right\rceil \geq \frac{cn}{N_i m} \geq \frac{cn}{N_i \log(N_1+N_2)} \geq c \left\lceil \frac{n}{N_i \log(N_1+N_2)} \right\rceil \quad (i=1,2). \tag{62}$$

Combining (61) and (62) proves the upper bound in (56).

Now we prove the lower bounds in (55)–(57). We use Lemma 1 and Proposition 3. We assume $n < \frac{1}{2}c(0)N_1N_2$, where $c(0)$ stands for the constant c_0 from Proposition 3. We start with (55) and assume first that $p \geq q$ and $u \leq v$. Setting $n_1 = \left\lceil \frac{1}{2}c(0)N_1N_2 \right\rceil - 1$, we have $n \leq n_1$ and

$$\frac{1}{4}c(0)N_1N_2 \leq n_1 < \frac{1}{2}c(0)N_1N_2,$$

therefore, by (34) and monotonicity of the n-th minimal errors

$$\begin{aligned} e_n^{\mathrm{ran}}\left(J^{N_1,N_2}, B_{L_p^{N_1}(L_u^{N_2})}, L_q^{N_1}(L_v^{N_2})\right) &\geq e_{n_1}^{\mathrm{ran}}\left(J^{N_1,N_2}, B_{L_p^{N_1}(L_u^{N_2})}, L_q^{N_1}(L_v^{N_2})\right) \\ \geq \frac{1}{2}e_{2n_1}^{\mathrm{avg}}(J^{N_1,N_2}, \mu^{(5)}_{2n_1,N_1,N_2}, L_q^{N_1}(L_v^{N_2})) &\geq c\left\lceil \frac{2n_1}{N_1}\right\rceil^{1/u-1/v} \geq cN_2^{1/u-1/v}, \end{aligned}$$

which is the lower bound of (55) for the case $(p \geq q) \wedge (u \leq v)$. If $(p \geq q) \wedge (u > v)$, the lower estimate of relation (55) follows from (33), while if $(p < q) \wedge (u \leq v)$, it is a consequence of (32). Relations (30) and (31) together give the lower bound in (56), and (35) implies the one in (57). $\square$

With c_0 from Theorem 1 and $N_1 = N_2 = \left\lfloor c_0^{-1/2} n^{1/2} \right\rfloor + 1$, we obtain

Corollary 1. *There are constants $c_1, c_2 > 0$ such that for each $n \in \mathbb{N}$ there exist $N_1, N_2 \in \mathbb{N}$ such that*

$$c_1 n^{1/2}(\log(n+1))^{-1} \leq \frac{e_n^{\mathrm{ran-non}}(J^{N_1,N_2}, B_{L_1^{N_1}(L_\infty^{N_2})}, L_\infty^{N_1}(L_1^{N_2}))}{e_n^{\mathrm{ran}}(J^{N_1,N_2}, B_{L_1^{N_1}(L_\infty^{N_2})}, L_\infty^{N_1}(L_1^{N_2}))} \leq c_2 n^{1/2}.$$

It is not known though if the exponent 1/2 is the largest possible among all linear problems. More precisely, is $\sup\Gamma > 1/2$, where

$$\Gamma = \left\{\gamma > 0 : \exists c > 0\, \forall n \in \mathbb{N}\, \exists \text{ a linear problem } \mathcal{P}_n \text{ with } \frac{e_n^{\mathrm{ran-non}}(\mathcal{P}_n)}{e_n^{\mathrm{ran}}(\mathcal{P}_n)} \geq cn^\gamma \right\}?$$

References

[1] S. Gal, C. A. Micchelli, Optimal sequential and non-sequential procedures for evaluating a functional, Appl. Anal. 10 (1980), 105-120.

[2] S. Heinrich, Quantum summation with an application to integration, Journal of Complexity 18 (2002), 1-50.

[3] S. Heinrich, Monte Carlo approximation of weakly singular integral operators, J. Complexity 22 (2006), 192–219.

[4] S. Heinrich, The randomized information complexity of elliptic PDE, J. Complexity 22 (2006), 220–249.

[5] S. Heinrich, On the complexity of computing the L_q norm, J. Complexity 49 (2018), 1-26

[6] S. Heinrich, Complexity of stochastic integration in Sobolev classes, J. Math. Anal. Appl. 476 (2019), 177-195.

[7] S. Heinrich, Randomized complexity of parametric integration and the role of adaption I. Finite dimensional case, preprint, see https://www.uni-kl.de/AG-Heinrich/Publications.html

[8] E. Novak, Deterministic and Stochastic Error Bounds in Numerical Analysis, Lecture Notes in Mathematics 1349, Springer-Verlag, 1988.

[9] E. Novak, On the power of adaption, J. Complexity 12 (1996), 199-237.

[10] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume 1, Linear Information, European Math. Soc., Zürich, 2008.

[11] J. F. Traub, G. W. Wasilkowski, H. Woźniakowski, Information-Based Complexity, Academic Press, 1988.

[12] J. F. Traub, H. Woźniakowski, A General Theory of Optimal Algorithms, Academic Press, New York, 1980.