The Randomized Information Complexity of Elliptic PDE

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Abstract

We study the information complexity in the randomized setting of solving a general elliptic PDE of order $2m$ in a smooth, bounded domain $Q \subset \mathbb{R}^d$ with smooth coefficients and homogeneous boundary conditions. The solution is sought on a smooth submanifold $M \subseteq Q$ of dimension $0 \le d_1 \le d$, the right hand side is supposed to be in $C^r(Q)$, the error is measured in the $L_\infty(M)$ norm. We show that the n-th minimal error is (up to logarithmic factors) of order

$$n^{-\min((r+2m)/d_1,\, r/d+1/2)}.$$

For comparison, in the deterministic setting the n-th minimal error is of order $n^{-r/d}$, for all d_1.

1 Introduction

In this paper we are concerned with the complexity of solving elliptic partial differential equations. We shall mainly deal with the randomized setting. For the complexity of elliptic equations in the deterministic setting, we refer to [19, 21, 22, 4, 2, 3], and the references therein. The complexity of certain parabolic problems was investigated in [18] and [14] in the deterministic, and in [13], [17] in the randomized setting. The complexity of elliptic problems in the randomized setting has not been studied before. This is the main aim of the present paper.

Based on results about the randomized approximation of weakly singular operators [9], and the Green's function representation of solutions of elliptic partial differential equations [12], we determine the information complexity of solving general elliptic problems with homogeneous boundary conditions. While in previous work on the deterministic setting of elliptic problems only the global problem is considered (that is, one seeks to approximate the solution in the whole domain), the above mentioned work on parabolic problems considers only the solution in a single point (the local problem). This is connected with the path integration approach and related representations.

Here we analyze a whole range of problems — the solution being sought on a smooth, d_1-dimensional submanifold, including the local problem for $d_1 = 0$ and the global one for $d_1 = d$. We are concerned with the information complexity, that means, determining the minimal number of function value calls any algorithm has to invoke in order to reach a certain error. Equivalently, we study the minimal error among all possible algorithms making not more than a given number of function calls.

This approach gives strong lower bounds. The upper bounds can be considered as approximation theoretic bounds, or benchmarks for concrete, implementable algorithms. For we only count information calls, all other operations are considered as free. This corresponds to the query complexity, studied in the quantum setting in [13], and to the approach taken in [2, 3] for the deterministic setting. In the papers [18], [14], [13], [17] precomputing is considered free, which is essentially equivalent to our assumption. For a special case of an elliptic problem, a fully implementable algorithm with the number of operations being of the optimal order was presented in [9].

We consider adaptive randomized algorithms. For our analysis we need a number of technical results on n-th minimal errors, such as reduction, additivity, and multiplicativity. Although such properties are sometimes applied in simple situations in an informal way, and a first formal approach was given for nonadaptive algorithms in [15], there are no rigorous general results on adaptive randomized algorithms in the literature. We therefore have chosen to present and prove the needed results in full rigor. It turned out to be convenient to formulate a model of computation which is formally slightly more general, but, in fact, equivalent to the usual one used in information complexity analysis.

The paper is organized as follows. In section 2 we describe the problem to be solved and state the main result. The needed results about n-th minimal errors are derived in section 3. Section 4 contains the proof of the upper bound of the main result, while the proof of the lower bound is given in section 5.

Basic notation, facts and background on information-based complexity theory – the framework in which we carry out our investigations – can be found in [19, 16, 6].

The complexity of solving elliptic PDE in the quantum model of computation will be the topic of a subsequent paper [10].

2 Preliminaries and the Result

Let $d \in \mathbb{N}$ (throughout the paper $\mathbb{N}$ means $\{1, 2, \dots\}$, while $\mathbb{N}_0$ means $\mathbb{N} \cup \{0\}$). Let $Q \subset \mathbb{R}^d$ be the closure of a non-empty open bounded set. The boundary of Q is denoted by ∂Q, the interior by Q^0. Let $C(Q)$ be the space of continuous complex-valued functions on Q, equipped with the supremum norm, and let $C^r(Q)$ for $r \in \mathbb{N}$ be the space of continuous complex-valued functions on Q which are r-times continuously differentiable in Q^0, and whose partial derivatives up to order r have continuous extensions to Q. The norm on $C^r(Q)$ is defined as

$$\|f\|_{C^r(Q)} = \max_{|\alpha| \le r} \sup_{x \in Q} |D^\alpha f(x)|.$$

For a normed space G the unit ball $\{g \in G : \|g\| \le 1\}$ is denoted by B_G. A mapping $\Phi : Q \to \mathbb{R}^d$ is called a C^∞ diffeomorphism if there is an open set $U \subseteq \mathbb{R}^d$ with $Q \subset U$, and a mapping $\Psi : U \to \mathbb{R}^d$ such that $\Psi|_Q = \Phi$, Ψ is injective, infinitely often differentiable and $\det(\mathcal{J}_\Psi(y)) \neq 0$ for all $y \in U$, where $\mathcal{J}_\Phi$ is the Jacobian of Φ. (It follows that $\Psi(U)$ is open and Ψ^{-1} is infinitely differentiable on $\Psi(U)$.) Denote

$$\begin{aligned} W' &= [-1, 1]^d \\ W'_+ &= \{y = (y_1, \dots, y_d) \in W' : y_1 \ge 0\}. \end{aligned}$$

We say that Q is a C^∞ domain, if for each point $x \in \partial Q$ there is a closed neighbourhood U_x of x in $\mathbb{R}^d$ and a C^∞ diffeomorphism Φ_x from W' onto U_x with

$$\Phi_x(0) = x \tag{1}$$

$$\Phi_x^{-1}(U_x \cap Q) = W'_+. \tag{2}$$

Let $d_1 \in \mathbb{N}_0$, $0 \le d_1 \le d$. A subset M of a C^∞ domain Q is called a d_1-dimensional C^∞ submanifold of Q, if M is closed, for each point $x \in M \cap \partial Q$ there is a closed neighbourhood U_x of x in $\mathbb{R}^d$ and a C^∞ diffeomorphism Φ_x from W' onto U_x with the properties (1), (2), and

$$\Phi_x^{-1}(U_x \cap M) = \{y = (y_1, \dots, y_d) \in W'_+ : y_{d_1+1} = \dots = y_d = 0\}, \tag{3}$$

and moreover, for each point $x \in M \cap Q^0$ there is a closed neighbourhood $U_x \subset Q^0$ of x and a C^∞ diffeomorphism Φ_x of W' onto U_x such that

$$\begin{aligned} \Phi_x(0) &= x & (4)\\ \Phi_x^{-1}(U_x \cap M) &= \{y = (y_1, \ldots, y_d) \in W' \,:\, y_{d_1+1} = \cdots = y_d = 0\}. & (5) \end{aligned}$$

It follows that if $d_1 = 0$, then M is just any finite set of points of Q, and if $d_1 = d$, then M is the union of connected components of Q (it follows from the assumptions above that there are only finitely many such components).

Let $d \geq 2$, $m \in \mathbb{N}$ and let $\mathscr{L}$ be an elliptic differential operator of order $2m$ on a C^∞ domain $Q \subset \mathbb{R}^d$, that is

$$\mathscr{L}u = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u(x), \tag{6}$$

with boundary operators

$$\mathscr{B}_j u = \sum_{|\alpha| \leq m_j} b_{j\alpha}(x) D^\alpha u(x), \tag{7}$$

where $j = 1, \ldots, m$, $m_j \leq 2m - 1$, $a_\alpha \in C^\infty(Q)$ and $b_{j\alpha} \in C^\infty(\partial Q)$. Here $C^\infty(Q) = \cap_{s \in \mathbb{N}} C^s(Q)$, and $b \in C^\infty(\partial Q)$ means that for all $x \in \partial Q$ we have $b \circ \Phi_x|_{W_1'} \in C^\infty(W_1')$, with $W_1' = \{0\} \times [-1, 1]^{d-1}$. Define

$$\begin{aligned} a(x, \xi) &:= \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha \quad (x \in Q,\ \xi \in \mathbb{R}^d)\\ b_j(x, \xi) &:= \sum_{|\alpha| = m_j} b_{j\alpha}(x) \xi^\alpha \quad (x \in \partial Q,\ \xi \in \mathbb{R}^d,\ j = 1, \ldots, m). \end{aligned}$$

We assume that $\mathscr{L}$ satisfies the ellipticity condition:

$$a(x, \xi) \neq 0 \quad (x \in Q,\ \xi \in \mathbb{R}^d \setminus \{0\})$$

and for each pair of linearly independent vectors $\xi, \eta \in \mathbb{R}^d$ the polynomial $a(x, \xi + \tau\eta)$ has exactly m roots τ_i^+ ($i = 1, \ldots, m$) with positive imaginary part. Denote

$$a^+(x, \xi, \eta, \tau) = \prod_{i=1}^m (\tau - \tau_i^+).$$

We also assume that the complementarity condition is satisfied: For each $x \in \partial Q$ and each pair of vectors $\xi_x, \nu_x \in \mathbb{R}^d \setminus \{0\}$ with ξ_x being tangent to ∂Q at x and ν_x being orthogonal to the tangent hyperplane at x, the set of polynomials $b_j(x, \xi_x + \tau\nu_x)$ $(j = 1, \dots, m)$ is linearly independent modulo $a^+(x, \xi_x, \nu_x, \tau)$. (These are the assumptions made in Krasovskij [12], and formulated in detail in [11]. The same conditions, except for the restriction $m_j \le 2m-1$, are imposed in [1]. In both papers instead of infinite smoothness a certain finite smoothness of the coefficients and boundary is assumed.) We consider the homogeneous boundary value problem

$$\mathscr{L}u(x) = f(x) \quad (x \in Q^0) \tag{8}$$

$$\mathscr{B}_j u(x) = 0 \quad (x \in \partial Q). \tag{9}$$

Finally we assume, as in [12], p. 963, that there is a κ_0 with $0 < \kappa_0 < 1$ such that for all f in the Hölder space $C^{\kappa_0}(Q)$ the (classical) solution u exists and is unique.

Let M be as defined above – a smooth submanifold of Q of dimension d_1, where $0 \le d_1 \le d$. If $d_1 = 0$, we assume $M = \{x\}$, where x is any inner point of Q. Let $r \in \mathbb{N}$,

$$F = B_{C^r(Q)}, \qquad G = L_\infty(M),$$

and define the solution operator S as follows:

$$S : F \to G \qquad Sf = u|_M,$$

where u is the solution of (8), (9). Thus, we seek to find an approximation of the solution of the boundary value problem on a d_1-dimensional submanifold M of the domain Q, for right-hand sides belonging to $B_{C^r(Q)}$, the error being measured in the norm of $L_\infty(M)$. We admit point value information of the function f and its derivatives, that is, the set of admissible information functionals is

$$\Lambda = \{\delta_x^\alpha \,:\, x \in Q,\ |\alpha| \le r\}$$

where $\delta_x^\alpha(f) = D^\alpha f(x)$. We shall study the complexity of approximating S. Let $e_n^{\mathrm{det}}(S, F)$ and $e_n^{\mathrm{ran}}(S, F)$ be the n-th minimal deterministic and randomized errors, that is, the minimal error over F among all deterministic, respectively randomized algorithms that use not more than n information values (see the next section for the definitions).

For $\sigma \in \mathbb{R}$ with $-d < \sigma < +\infty$ let

$$\kappa(\sigma) = \begin{cases} 0 & \text{if } \frac{d+\sigma}{d_1} > \frac{1}{2} \\ \frac{3}{2} & \text{if } \frac{d+\sigma}{d_1} = \frac{1}{2} \\ \frac{d+\sigma}{d_1} & \text{if } \frac{d+\sigma}{d_1} < \frac{1}{2} \quad \text{and} \quad d_1 = d \\ \frac{3}{2} & \text{if } \frac{d+\sigma}{d_1} < \frac{1}{2}, \quad d_1 < d, \quad \text{and} \quad \frac{r+d+\sigma}{d_1} \ne \frac{r}{d} + \frac{1}{2} \\ \frac{r}{d} + 3 & \text{if } \frac{d+\sigma}{d_1} < \frac{1}{2}, \quad d_1 < d, \quad \text{and} \quad \frac{r+d+\sigma}{d_1} = \frac{r}{d} + \frac{1}{2}. \end{cases} \tag{10}$$

The case $d_1 = 0$ is included in (10) and in the theorem below by interpreting $\frac{d+\sigma}{d_1} = +\infty$.

Theorem 1. *There are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ with $n \geq 2$,*

$$c_1 n^{-\frac{r}{d}} \leq e_n^{\text{det}}(S, F) \leq c_2 n^{-\frac{r}{d}} \tag{11}$$

and

$$\begin{aligned} c_1 n^{-\min\left(\frac{r+2m}{d_1},\, \frac{r}{d}+\frac{1}{2}\right)} &\leq e_n^{\text{ran}}(S, F) \\ &\leq c_2 n^{-\min\left(\frac{r+2m}{d_1},\, \frac{r}{d}+\frac{1}{2}\right)} (\log n)^{\kappa(2m-d)}. \end{aligned} \tag{12}$$

Although we are mainly interested in the randomized setting, the deterministic case is included into the statements for various reasons: Such results have not been formulated for the function spaces we consider here (usually at least the target space of S is a Hilbert space). Moreover, no lower bounds for the case of submanifolds have been considered. And finally, it is done for the sake of comparison.

3 Some General Properties of n-th Minimal Errors

In this section we derive some general results on minimal error quantities like reduction, additivity and multiplicativity, which will be needed in the sequel. For background on information-based complexity theory, we refer to [19], the specific formalism used here can be found in section 4 of [9]. We briefly recall the basic notions. We consider a general numerical problem, given by a tuple $\mathcal{P} = (F, G, S, K, \Lambda)$, where F is a non-empty set, G a normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, S a mapping from F to G, K a non-empty set and Λ a non-empty set of mappings from F to K. Let $k^* = K$ (this choice just guarantees that $k^* \notin K$), and define the zero-th power of K as $K^0 = \{k^*\}$. We consider $f \in F$ also as a function on Λ with values in K by setting $f(\lambda) := \lambda(f)$. Let $\mathcal{F}(\Lambda, K)$ denote the set of all mappings from Λ to K. Let $m, n \in \mathbb{N}_0$ and define the concatenation operation $\oplus : K^m \times K^n \to K^{m+n}$ as follows: For $p = (k_1, \dots, k_m) \in K^m$ and $q = (l_1, \dots, l_n) \in K^n$ with $m, n \in \mathbb{N}$ we put

$$p \oplus q = (k_1, \dots, k_m, l_1, \dots, l_n).$$

If $m = 0$ or $n = 0$, we define

$$p \oplus k^* = k^* \oplus p = p.$$

A deterministic algorithm A for $\mathcal{P}$ is a tuple

$$A = ((L_i)_{i=1}^{\infty}, (\tau_i)_{i=0}^{\infty}, (\varphi_i)_{i=0}^{\infty})$$

where for each i,

$$\begin{aligned} L_i &: K^{i-1} \to \Lambda \\ \tau_i &: K^i \to \{0,1\} \\ \varphi_i &: K^i \to G \end{aligned}$$

are arbitrary mappings. Let $\mathcal{A}^{\mathrm{det}}(\mathcal{P})$ (or shortly $\mathcal{A}^{\mathrm{det}}$) denote the set of all deterministic algorithms for $\mathcal{P}$. For $f \in \mathcal{F}(\Lambda, K)$ and $A \in \mathcal{A}^{\mathrm{det}}$, the computational sequence of A at input f, is defined as follows:

$$\begin{aligned} z_0 &= k^* \\ z_i &= (f(L_1(z_0)), \ldots, f(L_i(z_{i-1}))) \quad (i \geq 1). \end{aligned}$$

The cardinality $\mathrm{card}(A, f)$ of A at input f is the first integer $n \geq 0$ with $\tau_n(z_n) = 1$, and $\mathrm{card}(A, f) = +\infty$ if there is no such n. Define

$$\mathrm{Dom}(A) = \{f \in \mathcal{F}(\Lambda, K) : \ \mathrm{card}(A, f) < \infty\}.$$

For $f \in \mathrm{Dom}(A)$ and $n = \mathrm{card}(A, f)$ we define the output $A(f)$ of algorithm A at input f as

$$A(f) = \varphi_n(z_n).$$

Define

$$\mathrm{card}(A, F) = \sup_{f \in F} \mathrm{card}(A, f),$$

and the error of A as

$$e(S, A, F) = \sup_{f \in F} \|S(f) - A(f)\|_G$$

if $F \subseteq \mathrm{Dom}(A)$, and $e(S, A, F) = +\infty$ otherwise. For $n \in \mathbb{N}_0$, the n-th deterministic minimal error is defined as

$$e_n^{\mathrm{det}}(S, F) = \inf\{e(S, A, F) : A \in \mathcal{A}^{\mathrm{det}}, \ \mathrm{card}(A, F) \leq n\}.$$

A randomized (or Monte Carlo) algorithm for $\mathcal{P}$

$$A = ((\Omega, \Sigma, \mu), \ (A_\omega)_{\omega \in \Omega}),$$

consists of a probability space (Ω, Σ, μ), and a family

$$A_\omega \in \mathcal{A}^{\det}(\mathcal{P}) \quad (\omega \in \Omega).$$

Let $\mathcal{A}^{\mathrm{ran}}(\mathcal{P})$, or shortly $\mathcal{A}^{\mathrm{ran}}$ be the class of all randomized algorithms for $\mathcal{P}$. For $A \in \mathcal{A}^{\mathrm{ran}}$ let $\mathrm{Dom}(A)$ be the set of all $f \in \mathcal{F}(\Lambda, K)$ such that $\mathrm{card}(A_\omega, f)$ is a measurable function of ω,

$$\mathrm{card}(A_\omega, f) < \infty \text{ for almost all } \omega \in \Omega,$$

and $A_\omega(f)$ is a G-valued random variable, that is, $A_\omega(f)$ is Borel measurable and there is a separable subspace G_0 of G (which may depend on f) such that

$$A_\omega(f) \in G_0 \text{ for almost all } \omega \in \Omega.$$

For $f \in \mathcal{F}(\Lambda, K)$, let

$$\mathrm{card}(A, f) = \int_\Omega \mathrm{card}(A_\omega, f) \, d\mu(\omega)$$

if $f \in \mathrm{Dom}(A)$ and $\mathrm{card}(A, f) = +\infty$ otherwise, and set

$$\mathrm{card}(A, F) = \sup_{f \in F} \mathrm{card}(A, f).$$

We define the error of $A \in \mathcal{A}^{\mathrm{ran}}$ by

$$e(S, A, F) = \sup_{f \in F} \int_\Omega \|S(f) - A_\omega(f)\|_G \, d\mu(\omega)$$

if $F \subseteq \mathrm{Dom}(A)$, and $e(S, A, F) = +\infty$ otherwise. For $n \in \mathbb{N}_0$ the n-th randomized minimal error is defined as

$$e_n^{\mathrm{ran}}(S, F) = \inf\{e(S, A, F) : A \in \mathcal{A}^{\mathrm{ran}}, \ \mathrm{card}(A, F) \le n\}.$$

To prove the needed general statements in a mathematically rigorous way turns out to be extremely cumbersome if we can rely only on the standard definition of a deterministic algorithm as given above. Therefore we present here a formally more general approach, which is, in fact, equivalent to the standard one, as we show below. On the other hand it is more convenient to work with since it allows to store auxiliary information, such as information about previous stages or intermediate results, needed later. A considerable part of this section is devoted to the deterministic setting, because this provides the basis for the randomized setting.

An extended deterministic algorithm A for $\mathcal{P}$ is a tuple

$$A = ((Z_i)_{i=0}^\infty, (L_i)_{i=1}^\infty, (U_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty, z_0)$$

where for each i, Z_i is a non-empty set,

$$\begin{aligned} L_i &: Z_{i-1} \to \Lambda \\ U_i &: Z_{i-1} \times K \to Z_i \\ \tau_i &: Z_i \to \{0, 1\} \\ \varphi_i &: Z_i \to G \end{aligned}$$

are any mappings, and $z_0 \in Z_0$. We call Z_i the state space of A at stage i. Given $f \in \mathcal{F}(\Lambda, K)$, we associate with it a sequence $(z_i)_{i=0}^\infty$, we call it again the computational sequence of A at input f, where z_0 is fixed by the above, $z_i \in Z_i$, and for $i \in \mathbb{N}_0$,

$$z_{i+1} = U_{i+1}(z_i, f(L_{i+1}(z_i))).$$

Let $\operatorname{card}(A, f)$ be the first integer $n \geq 0$ with $\tau_n(z_n) = 1$, and put $\operatorname{card}(A, f) = +\infty$ if there is no such n. Define

$$\operatorname{Dom}(A) := \{f \in \mathcal{F}(\Lambda, K) : \operatorname{card}(A, f) < \infty\}.$$

For $f \in \operatorname{Dom}(A)$ and $n = \operatorname{card}(A, f)$ we put

$$A(f) = \varphi_n(z_n).$$

Thus, an extended algorithm starts in the initial state z_0. It collects information about f (in an adaptive way): At stage i the state z_i represents the information computed so far, possibly including intermediate results. On this basis a $\lambda_{i+1} = L_{i+1}(z_i) \in \Lambda$ is determined, and z_{i+1} is obtained from z_i and the new function value $f(\lambda_{i+1})$. The functions τ_i decide, when the computation is terminated. Then the output element is determined from the information contained in the last state, hence, as a function of the queried values $(f(\lambda_i))_{i=1}^n$. If $\operatorname{card}(A, f) = \infty$, the computation is considered as going on forever and the output is undefined. Clearly, the standard definition of a deterministic algorithm given above corresponds to the special case $Z_i = K^i$ and $U_i(z, k) = z \oplus k \quad (z \in K^{i-1},\, k \in K)$.

It turns out that the formally more general definition we gave is, in fact, equivalent to the standard one. The precise formulation is contained in the lemma below:

Lemma 1. *For each extended deterministic algorithm $\widetilde{A}$ for $\mathcal{P}$ there is a deterministic algorithm $A \in \mathcal{A}^{\det}(\mathcal{P})$ such that for all $f \in \mathcal{F}(\Lambda, K)$*

$$\operatorname{card}(A, f) = \operatorname{card}(\widetilde{A}, f),$$

hence $\operatorname{Dom}(A) = \operatorname{Dom}(\widetilde{A})$*. Furthermore, for all* $f \in \operatorname{Dom}(A)$

$$A(f) = \widetilde{A}(f).$$

Proof. Let

$$\widetilde{A} = ((\widetilde{Z}_i)_{i=0}^{\infty}, (\widetilde{L}_i)_{i=1}^{\infty}, (\widetilde{U}_i)_{i=1}^{\infty}, (\widetilde{\tau}_i)_{i=0}^{\infty}, (\widetilde{\varphi}_i)_{i=0}^{\infty}, \widetilde{z}_0).$$

Define $\zeta_i : K^i \to \widetilde{Z}_i$ by

$$\begin{aligned} \zeta_0(k^*) &= \widetilde{z}_0, \\ \zeta_1(k) &= \widetilde{U}_1(\widetilde{z}_0, k) \\ \zeta_{i+1}(k_1, \ldots, k_{i+1}) &= \widetilde{U}_{i+1}(\zeta_i(k_1, \ldots, k_i), k_{i+1}) \quad (i \ge 1) \end{aligned}$$

and for $i \ge 0$

$$\begin{aligned} L_{i+1} &= \widetilde{L}_{i+1} \circ \zeta_i \\ \tau_i &= \widetilde{\tau}_i \circ \zeta_i \\ \varphi_i &= \widetilde{\varphi}_i \circ \zeta_i. \end{aligned}$$

Let $A = ((L_i)_{i=1}^{\infty}, (\tau_i)_{i=0}^{\infty}, (\varphi_i)_{i=0}^{\infty})$. Now fix $f \in \mathcal{F}(\Lambda, K)$. Let $(z_i)_{i=0}^{\infty}$ and $(\widetilde{z}_i)_{i=0}^{\infty}$ be the respective computational sequences of A and $\widetilde{A}$. We show by induction that

$$\widetilde{z}_i = \zeta_i(z_i). \tag{13}$$

For $i = 0$ this follows by the definition of ζ_0. Now assume (13) holds for a certain $i \ge 0$. Then

$$\widetilde{L}_{i+1}(\widetilde{z}_i) = \widetilde{L}_{i+1}(\zeta_i(z_i)) = L_{i+1}(z_i).$$

Hence,

$$\begin{aligned} \widetilde{z}_{i+1} &= \widetilde{U}_{i+1}(\widetilde{z}_i, f(\widetilde{L}_{i+1}(\widetilde{z}_i))) \\ &= \widetilde{U}_{i+1}(\zeta_i(z_i), f(L_{i+1}(z_i))) \\ &= \zeta_{i+1}(z_i \oplus f(L_{i+1}(z_i))) = \zeta_{i+1}(z_{i+1}). \end{aligned}$$

This proves (13). It follows that

$$\widetilde{\tau}_i(\widetilde{z}_i) = \widetilde{\tau}_i(\zeta_i(z_i)) = \tau_i(z_i),$$

consequently

$$\operatorname{card}(A,f)=\operatorname{card}(\widetilde{A},f).$$

If $n=\operatorname{card}(A,f)<\infty$, we thus get

$$\widetilde{A}(f)=\widetilde{\varphi}_n(\widetilde{z}_n)=\widetilde{\varphi}_n(\zeta_n(z_n))=\varphi_n(z_n)=A(f).$$

□

In complexity theory, an important ingredient for lower bound proofs is reduction. For continuous problems in the deterministic and randomized setting, reductions are usually applied on an informal basis. The only reference which contains a formal notion (called subordination there) is the habilitation thesis of Mathé [15]. In the present paper reductions are needed in many places, so we develop the required tools in a rigorous way. Our approach is inspired by [15] (but is somewhat more general, since it includes adaptive algorithms) and by recent work on the quantum setting (compare Lemma 1 and Corollary 1 of [7]).

Let $\widetilde{\mathcal{P}}=(\widetilde{F},\widetilde{G},\widetilde{S},\widetilde{K},\widetilde{\Lambda})$ be another numerical problem. Suppose we are in the following situation: We have an algorithm for problem $\widetilde{\mathcal{P}}$, and we want one for problem $\mathcal{P}$. Moreover, for each input $f\in F$ of problem $\mathcal{P}$ we can produce an input $R(f)$ for problem $\widetilde{\mathcal{P}}$ such that the solution $S(f)$ is obtained as the solution $\widetilde{S}(R(f))$, to which a mapping Ψ (symbolizing a certain computation) is applied. Furthermore, each information about $R(f)$ can be obtained from κ suitable informations about f and the application of a certain mapping. In this sense, problem $\mathcal{P}$ reduces to $\widetilde{\mathcal{P}}$.

We want to estimate the minimal error of S through the minimal error of $\widetilde{S}$. The result is given in the next proposition. Before we state it, let us formulate the precise assumptions.

Assume that $R:F\to\widetilde{F}$ is a mapping such that there exist a $\kappa\in\mathbb{N}$, mappings $\eta_j:\widetilde{\Lambda}\to\Lambda\quad(j=1,\dots,\kappa)$ and $\varrho:\widetilde{\Lambda}\times K^\kappa\to\widetilde{K}$ with

$$(R(f))(\widetilde{\lambda})=\varrho(\widetilde{\lambda},f(\eta_1(\widetilde{\lambda})),\dots,f(\eta_\kappa(\widetilde{\lambda}))) \tag{14}$$

for all $f\in F$ and $\widetilde{\lambda}\in\widetilde{\Lambda}$. Observe that (14) also defines a mapping $R:\mathcal{F}(\Lambda,K)\to\mathcal{F}(\widetilde{\Lambda},\widetilde{K})$, which we denote by the same symbol. Suppose that $\Psi:\widetilde{G}\to G$ is a Lipschitz mapping, that is, there is a constant $c\geq 0$ such that

$$\|\Psi(x)-\Psi(y)\|_G\leq c\,\|x-y\|_{\widetilde{G}}\quad\text{for all}\quad x,y\in\widetilde{G}.$$

The Lipschitz constant $\|\Psi\|_{\mathrm{Lip}}$ is the smallest constant c such that the relation above holds. Assume furthermore, that the solution operators S and $\widetilde{S}$

of $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are related in the following way:

$$S = \Psi \circ \widetilde{S} \circ R.$$

Proposition 1. *For all $n \in \mathbb{N}_0$,*

$$\begin{aligned} e_{\kappa n}^{\mathrm{det}}(S,F) &\le \|\Psi\|_{\mathrm{Lip}}\, e_n^{\mathrm{det}}(\widetilde{S},\widetilde{F}) \qquad (15)\\ e_{\kappa n}^{\mathrm{ran}}(S,F) &\le \|\Psi\|_{\mathrm{Lip}}\, e_n^{\mathrm{ran}}(\widetilde{S},\widetilde{F}). \qquad (16)\end{aligned}$$

Before we prove Proposition 1, we show how a deterministic algorithm for $\widetilde{\mathcal{P}}$ can be expressed by a suitable one for $\mathcal{P}$.

Lemma 2. *For each algorithm $\widetilde{A} \in \mathcal{A}^{\mathrm{det}}(\widetilde{\mathcal{P}})$ there is an algorithm $A \in \mathcal{A}^{\mathrm{det}}(\mathcal{P})$ such that for all $f \in \mathcal{F}(\Lambda, K)$*

$$\mathrm{card}(A,f) = \kappa\, \mathrm{card}(\widetilde{A}, R(f)), \qquad (17)$$

hence $f \in \mathrm{Dom}(A)$ iff $R(f) \in \mathrm{Dom}(\widetilde{A})$. Moreover, for all $f \in \mathrm{Dom}(A)$,

$$A(f) = \Psi \circ \widetilde{A} \circ R(f). \qquad (18)$$

Proof. Let $\widetilde{A} = ((\widetilde{L}_i)_{i=1}^{\infty}, (\widetilde{\tau}_i)_{i=0}^{\infty}, (\widetilde{\varphi}_i)_{i=0}^{\infty})$. Define an extended deterministic algorithm

$$A = ((Z_l)_{l=0}^{\infty}, (L_l)_{l=1}^{\infty}, (U_l)_{l=1}^{\infty}, (\tau_l)_{l=0}^{\infty}, (\varphi_l)_{l=0}^{\infty}, z_0)$$

by setting for

$$\begin{gathered} l = \kappa i + j \quad i \ge 0,\ 0 \le j < \kappa, \\ Z_l = K^j \times \widetilde{K}^i, \text{ thus } z_0 = (k^*, \widetilde{k}^*), \end{gathered}$$

(in the first component we store, step by step, the κ informations on $\mathcal{P}$ needed to compute one information on $\widetilde{\mathcal{P}}$, in the second component we simulate the computation of $\widetilde{A}$), and for $(q,\widetilde{q}) \in K^j \times \widetilde{K}^i$,

$$\begin{aligned} L_{l+1}(q,\widetilde{q}) &= \eta_{j+1}(\widetilde{L}_{i+1}(\widetilde{q})) \\ U_{l+1}(q,\widetilde{q},k) &= (q \oplus k, \widetilde{q}) \quad \text{if } 0 \le j < \kappa - 1 \\ U_{l+1}(q,\widetilde{q},k) &= (k^*, \widetilde{q} \oplus \varrho(\widetilde{L}_{i+1}(\widetilde{q}), q \oplus k)) \quad \text{if } j = \kappa - 1 \\ \tau_l(q,\widetilde{q}) &= \widetilde{\tau}_i(\widetilde{q}) \\ \varphi_l(q,\widetilde{q}) &= \Psi \circ \widetilde{\varphi}_i(\widetilde{q}). \end{aligned}$$

Let $f \in \mathcal{F}(\Lambda, K)$ and let $(\widetilde{z}_i)_{i=0}^{\infty}$ be the computational sequence of $\widetilde{A}$ at input $R(f)$, and $(z_l)_{l=0}^{\infty}$ that of A at f. Let $z_l = (q_l, \widetilde{q}_l)$. We show by induction that

$$\widetilde{q}_l = \widetilde{z}_i \quad \text{for } l = \kappa i + j,\ i \ge 0,\ 0 \le j < \kappa. \qquad (19)$$

Indeed, this holds by definition for $i = j = 0$. Now assume (19) holds for $l = \kappa i$ for a fixed $i \geq 0$. We shall show that then it also holds for $l = \kappa i + j + 1$ for all $0 \leq j < \kappa$. By assumption and definition we have

$$z_{\kappa i} = (k^*, \widetilde{z}_i).$$

It follows from the definition that

$$z_{\kappa i + j} = (q_{\kappa i + j}, \widetilde{z}_i) \quad (0 \leq j < \kappa).$$

Denote $\widetilde{L}_{i+1}(\widetilde{z}_i) = \widetilde{\lambda}$. Then

$$L_{\kappa i+j+1}(z_{\kappa i+j}) = L_{\kappa i+j+1}(q_{\kappa i+j}, \widetilde{z}_i) = \eta_{j+1}(\widetilde{L}_{i+1}(\widetilde{z}_i)) = \eta_{j+1}(\widetilde{\lambda}).$$

Consequently

$$z_{\kappa i+j} = (f(\eta_1(\widetilde{\lambda})), \ldots, f(\eta_j(\widetilde{\lambda})), \widetilde{z}_i) \quad (1 \leq j < \kappa),$$

and therefore,

$$\begin{aligned} z_{\kappa i+\kappa} &= U_{\kappa i+\kappa}(z_{\kappa i+\kappa-1}, f(L_{\kappa i+\kappa}(z_{\kappa i+\kappa-1}))) \\ &= U_{\kappa i+\kappa}((f(\eta_1(\widetilde{\lambda})), \ldots, f(\eta_{\kappa-1}(\widetilde{\lambda})), \widetilde{z}_i), f(\eta_\kappa(\widetilde{\lambda}))) \\ &= (k^*, \widetilde{z}_i \oplus \varrho(\widetilde{\lambda}, f(\eta_1(\widetilde{\lambda})), \ldots, f(\eta_\kappa(\widetilde{\lambda})))) \\ &= (k^*, \widetilde{z}_i \oplus (R(f))(\widetilde{\lambda})) \\ &= (k^*, \widetilde{z}_i \oplus (R(f))(\widetilde{L}_{i+1}(\widetilde{z}_i))) = (k^*, \widetilde{z}_{i+1})). \end{aligned}$$

This proves (19). It follows that

$$\tau_l(z_l) = \widetilde{\tau}_i(\widetilde{q}_l) = \widetilde{\tau}_i(\widetilde{z}_i),$$

hence, letting $n := \mathrm{card}(A, f)$ and $\widetilde{n} := \mathrm{card}(\widetilde{A}, R(f))$, we have

$$n = \kappa \widetilde{n},$$

therefore $n < \infty$ iff $\widetilde{n} < \infty$, and in this case

$$\varphi_n(z_n) = \Psi \circ \widetilde{\varphi}_{\widetilde{n}}(\widetilde{z}_{\widetilde{n}}),$$

which gives

$$A(f) = \Psi \circ \widetilde{A} \circ R(f).$$

By Lemma 1, we can replace the extended deterministic algorithm A by a deterministic algorithm with the same properties (17) and (18), which yields the result. ☐

Proof of Proposition 1. Let $\widetilde{A} \in \widetilde{\mathcal{A}}^{\rm det}(\widetilde{\mathcal{P}})$ be any deterministic algorithm for $\widetilde{\mathcal{P}}$ with $\mathrm{card}(\widetilde{A}, \widetilde{F}) \le n$. Let $A \in \mathcal{A}^{\rm det}(\mathcal{P})$ be as obtained from $\widetilde{A}$ in Lemma 2. It follows from (17) and (18) that

$$\mathrm{card}(A, F) = \kappa\, \mathrm{card}(\widetilde{A}, R(F)) \le \kappa\, \mathrm{card}(\widetilde{A}, \widetilde{F}) \le \kappa n,$$

$$\begin{aligned} e(S, A, F) &= \sup_{f \in F} \|S(f) - A(f)\|_G \\ &= \sup_{f \in F} \|\Psi \circ \widetilde{S} \circ R(f) - \Psi \circ \widetilde{A} \circ R(f)\|_G \\ &\le \|\Psi\|_{\rm Lip} \sup_{f \in F} \|\widetilde{S} \circ R(f) - \widetilde{A} \circ R(f)\|_{\widetilde{G}} \\ &= \|\Psi\|_{\rm Lip}\, e(\widetilde{S}, \widetilde{A}, R(F)) \le \|\Psi\|_{\rm Lip}\, e(\widetilde{S}, \widetilde{A}, \widetilde{F}), \end{aligned}$$

and (15) follows. Now let

$$\widetilde{A} = ((\Omega, \Sigma, \mu),\ (\widetilde{A}_\omega)_{\omega \in \Omega}),$$

be a randomized algorithm for $\widetilde{\mathcal{P}}$ with $\mathrm{card}(\widetilde{A}, \widetilde{F}) \le n$. Hence $\widetilde{F} \subseteq \mathrm{Dom}(\widetilde{A})$. For each $\omega \in \Omega$, let A_ω be derived from $\widetilde{A}_\omega$ according to Lemma 2 and set

$$A = ((\Omega, \Sigma, \mu),\ (A_\omega)_{\omega \in \Omega}).$$

Then it follows from (17), (18) and the Lipschitz property of Ψ that $R(f) \in \mathrm{Dom}(\widetilde{A})$ implies $f \in \mathrm{Dom}(A)$ (compare the definition of a randomized algorithm given above). Moreover,

$$\mathrm{card}(A, f) = \kappa\, \mathrm{card}(\widetilde{A}, R(f)),$$

hence

$$\mathrm{card}(A, F) = \kappa\, \mathrm{card}(\widetilde{A}, R(F)) \le \kappa\, \mathrm{card}(\widetilde{A}, \widetilde{F}) \le \kappa n, \tag{20}$$

and

$$\begin{aligned} e(S, A, F) &= \sup_{f \in F} \int_\Omega \|S(f) - A_\omega(f)\|_G \; d\mu(\omega) \\ &= \sup_{f \in F} \int_\Omega \|\Psi \circ \widetilde{S} \circ R(f) - \Psi \circ \widetilde{A}_\omega \circ R(f)\|_G \; d\mu(\omega) \\ &\le \|\Psi\|_{\rm Lip} \sup_{f \in F} \int_\Omega \|\widetilde{S} \circ R(f) - \widetilde{A}_\omega \circ R(f))\|_{\widetilde{G}} \; d\mu(\omega) \\ &= \|\Psi\|_{\rm Lip}\, e(\widetilde{S}, \widetilde{A}, R(F)) \le \|\Psi\|_{\rm Lip}\, e(\widetilde{S}, \widetilde{A}, \widetilde{F}), \end{aligned}$$

which together with (20) implies (16) and completes the proof of Lemma 2. □

Next we establish additivity properties of the minimal error quantities.

Proposition 2. *Let $p \in \mathbb{N}$ and let $S_l : F \to G$ $(l = 1, \ldots, p)$ be mappings. Define $S : F \to G$ by $S(f) = \sum_{l=1}^p S_l(f) \quad (f \in F)$. Let $n_1, \ldots, n_p \in \mathbb{N}_0$ and put $n = \sum_{l=1}^p n_l$. Then*

$$
\begin{aligned}
e_n^{\det}(S, F) &\leq \sum_{l=1}^p e_{n_l}^{\det}(S_l, F), \\
e_n^{\mathrm{ran}}(S, F) &\leq \sum_{l=1}^p e_{n_l}^{\mathrm{ran}}(S_l, F).
\end{aligned}
$$

For the proof we need

Lemma 3. *Let $A_l \in \mathcal{A}^{\det}(\mathcal{P})$ for $l = 1, 2$. Then there is an algorithm $A \in \mathcal{A}^{\det}(\mathcal{P})$ such that for all $f \in \mathcal{F}(\Lambda, K)$*

$$
\mathrm{card}(A, f) = \mathrm{card}(A_1, f) + \mathrm{card}(A_2, f), \tag{21}
$$

consequently, $\mathrm{Dom}(A) = \mathrm{Dom}(A_1) \cap \mathrm{Dom}(A_2)$. *Moreover, for all $f \in$* $\mathrm{Dom}(A)$

$$
A(f) = A_1(f) + A_2(f). \tag{22}
$$

Proof. Let

$$
A_l = (L_{l,i})_{i=1}^\infty, (\tau_{l,i})_{i=0}^\infty, (\varphi_{l,i})_{i=0}^\infty) \quad (l = 1, 2).
$$

To define an extended deterministic algorithm A, put

$$
Z_i = K^i \times \{0, \ldots, i\} \times \{0, 1\}
$$

(we arrange a step counting component and a control bit). We shall first compute the information for A_1, then for A_2. The control bit is 0 while we are dealing with A_1, afterwards it is set to 1. The counting stops when the computation of A_1 is finished (and thus the counting variable shows how many informations we computed for A_1). Here are the formal details: Put

$$
z_0 = \begin{cases} (k^*, 0, 0) & \text{if} \quad \tau_{1,0}(k^*) = 0 \\ (k^*, 0, 1) & \text{if} \quad \tau_{1,0}(k^*) = 1. \end{cases}
$$

Let $z = (q, m, b) \in Z_i$. We represent q as $q = q_1 \oplus q_2$ with $q_1 \in K^m$ and

$q_2 \in K^{i-m}$. Define

$$\begin{aligned} L_{i+1}(z) &= \begin{cases} L_{1,i+1}(q) & \text{if} \quad b=0 \\ L_{2,i+1-m}(q_2) & \text{if} \quad b=1, \end{cases} \\ U_{i+1}(z,k) &= \begin{cases} (q\oplus k, m+1, 0) & \text{if} \quad b=0 \quad \text{and} \quad \tau_{1,i+1}(q\oplus k)=0 \\ (q\oplus k, m+1, 1) & \text{if} \quad b=0 \quad \text{and} \quad \tau_{1,i+1}(q\oplus k)=1 \\ (q\oplus k, m, 1) & \text{if} \quad b=1, \end{cases} \\ \tau_i(z) &= \begin{cases} 0 & \text{if} \quad b=0 \\ \tau_{2,i-m}(q_2) & \text{if} \quad b=1, \end{cases} \\ \varphi_i(z) &= \begin{cases} 0 & \text{if} \quad b=0 \\ \varphi_{1,m}(q_1)+\varphi_{2,i-m}(q_2) & \text{if} \quad b=1. \end{cases} \end{aligned}$$

We put

$$A = ((Z_i)_{i=0}^\infty, (L_i)_{i=1}^\infty, (U_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty, z_0).$$

Let $(z_{l,i})_{i=0}^\infty$ be the computational sequence of A_l at input $f \in \mathcal{F}(\Lambda, K)$ ($l = 1,2$), and let $(z_i)_{i=0}^\infty$ be the respective one for A. Denote $\operatorname{card}(A_l, f) = n_l$, and $\lambda_{l,i} = L_{l,i}(z_{l,i-1})$ for $i \geq 1$ and $l = 1,2$. We show that

$$z_i = \begin{cases} (z_{1,i}, i, 0) & \text{if} \quad i < n_1 \\ (z_{1,n_1} \oplus z_{2,i-n_1}, n_1, 1) & \text{if} \quad i \geq n_1. \end{cases} \tag{23}$$

This holds for $i = 0$ by definition. Now assume it holds for some $i \geq 0$. We show it for $i+1$. First we suppose $i < n_1$. By assumption, $z_i = (z_{1,i}, i, 0)$. Therefore,

$$\begin{aligned} L_{i+1}(z_i) &= L_{1,i+1}(z_{1,i}) = \lambda_{1,i+1} \\ z_{1,i} \oplus f(\lambda_{1,i+1}) &= z_{1,i+1} \\ \tau_{1,i+1}(z_{1,i} \oplus f(\lambda_{1,i+1})) &= \tau_{1,i+1}(z_{1,i+1}) = \begin{cases} 0 & \text{if} \quad i+1 < n_1 \\ 1 & \text{if} \quad i+1 = n_1. \end{cases} \end{aligned}$$

Thus, if $i+1 < n_1$

$$z_{i+1} = U_{i+1}(z_i, f(L_{i+1}(z_i))) = (z_{1,i} \oplus f(\lambda_{1,i+1}), i+1, 0) = (z_{1,i+1}, i+1, 0).$$

If $i+1 = n_1$, we have

$$\begin{aligned} z_{i+1} &= U_{i+1}(z_i, f(L_{i+1}(z_i))) = (z_{1,i} \oplus f(\lambda_{1,i+1}), i+1, 1) \\ &= (z_{1,i+1}, i+1, 1) = (z_{1,n_1} \oplus z_{2,i+1-n_1}, n_1, 1). \end{aligned}$$

Now suppose $i \geq n_1$. By assumption, $z_i = (z_{1,n_1} \oplus z_{2,i-n_1}, n_1, 1)$. Consequently,

$$\begin{aligned} L_{i+1}(z_i) &= L_{2,i+1-n_1}(z_{2,i-n_1}) = \lambda_{2,i+1-n_1} \\ z_{1,n_1} \oplus z_{2,i-n_1} \oplus f(\lambda_{2,i+1-n_1}) &= z_{1,n_1} \oplus z_{2,i+1-n_1}, \end{aligned}$$

and therefore

$$
\begin{aligned}
z_{i+1} &= U_{i+1}(z_i, f(L_{i+1}(z_i))) = (z_{1,n_1} \oplus z_{2,i-n_1} \oplus f(\lambda_{2,i+1-n_1}), n_1, 1) \\
&= (z_{1,n_1} \oplus z_{2,i+1-n_1}, n_1, 1).
\end{aligned}
$$

This proves (23). From this and the definition of τ_i we get

$$
\begin{aligned}
\tau_i(z_i) &= 0 \quad \text{if} \quad i < n_1 + n_2 \\
\tau_i(z_i) &= 1 \quad \text{if} \quad i = n_1 + n_2.
\end{aligned}
$$

Consequently,

$$\operatorname{card}(A, f) = n_1 + n_2 = \operatorname{card}(A_1, f) + \operatorname{card}(A_2, f).$$

This implies

$$\operatorname{Dom}(A) = \operatorname{Dom}(A_1) \cap \operatorname{Dom}(A_2).$$

Now assume $f \in \operatorname{Dom}(A_1) \cap \operatorname{Dom}(A_2)$. Setting $n = \operatorname{card}(A, f)$, we obtain

$$z_n = (z_{1,n_1} \oplus z_{2,n_2}, n_1, 1),$$

hence,

$$A(f) = \varphi_n(z_n) = \varphi_{1,n_1}(z_{1,n_1}) + \varphi_{2,n_2}(z_{2,n_2}) = A_1(f) + A_2(f).$$

The proof is completed by an application of Lemma 1. □

Proof of Proposition 2. Both the deterministic and randomized case follow by induction, once we show the case $p = 2$. Let $\delta > 0$ be arbitrary. First consider the deterministic setting. Let $A_l \in \mathcal{A}^{\det}$ $(l = 1, 2)$ be algorithms with

$$\operatorname{card}(A_l, F) \le n_l$$

and

$$e(S_l, A_l, F) = \sup_{f \in F} \|S_l(f) - A_l(f)\| \le e_{n_l}^{\det}(S_l, F) + \delta.$$

By Lemma 3 there is an $A \in \mathcal{A}^{\det}$ with

$$\operatorname{card}(A, F) \le n_1 + n_2$$

and

$$A(f) = A_1(f) + A_2(f) \quad (f \in F).$$

Consequently

$$\begin{aligned} e^{\mathrm{det}}_{n_1+n_2}(S,F) &\le e(S,A,F) = \sup_{f\in F}\|S(f)-A(f)\| \\ &= \sup_{f\in F}\Big\|\sum_{l=1,2}(S_l(f)-A_l(f))\Big\| \\ &\le \sup_{f\in F}\sum_{l=1,2}\|S_l(f)-A_l(f)\| \le \sum_{l=1,2} e^{\mathrm{det}}_{n_l}(S_l,F)+2\delta. \end{aligned}$$

Since $\delta > 0$ was arbitrary, this proves the deterministic case. In the randomized setting we argue similarly. Let $A_l \in \mathcal{A}^{\mathrm{ran}}$ $(l = 1, 2)$ be Monte Carlo algorithms

$$A_l = ((\Omega_l,\Sigma_l,\mu_l),\ (A_{l,\omega_l})_{\omega_l\in\Omega_l}),$$

with $\mathrm{card}(A_l, F) \le n_l$ and $e(S_l, A_l, F) \le e^{\mathrm{ran}}_{n_l}(S_l, F) + \delta$. By definition, this means

$$\sup_{f\in F}\int_{\Omega_l}\mathrm{card}(A_{\omega_l},f)d\mu_l(\omega_l) \le n_l \tag{24}$$

and

$$\sup_{f\in F}\int_{\Omega_l}\|S_l(f)-A_{l,\omega_l}(f)\|d\mu_l(\omega_l) \le e^{\mathrm{ran}}_{n_l}(S_l,F)+\delta. \tag{25}$$

Let (Ω, Σ, μ) be the product space

$$(\Omega_1\times\Omega_2,\Sigma_1\times\Sigma_2,\mu_1\times\mu_2).$$

Given $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, by Lemma 3 there is an $A_\omega \in \mathcal{A}^{\mathrm{det}}$ such that for all $f \in \mathcal{F}(\Lambda, K)$

$$\mathrm{card}(A_\omega,f) = \mathrm{card}(A_{1,\omega_1},f)+\mathrm{card}(A_{2,\omega_2},f), \tag{26}$$

and, if $\mathrm{card}(A_{1,\omega_1}, f) < \infty$ and $\mathrm{card}(A_{2,\omega_2}, f) < \infty$, then

$$A_\omega(f) = A_{1,\omega_1}(f)+A_{2,\omega_2}(f). \tag{27}$$

Define

$$A = ((\Omega,\Sigma,\mu),(A_\omega)_{\omega\in\Omega}).$$

Let $f \in F$. By (24) and (26), $\mathrm{card}(A_\omega, f)$ is measurable and almost surely finite. Moreover, by (27), $A_\omega(f)$ is a G-valued random variable (here the separability assumption is needed that was made in the definition of a randomized algorithm above). Thus, $F \subseteq \mathrm{Dom}(A)$. Moreover, by (26),

$$\begin{aligned} \mathrm{card}(A, F) &= \sup_{f \in F} \int_\Omega \mathrm{card}(A_\omega, f) d\mu(\omega) \\ &= \sup_{f \in F} \sum_{l=1,2} \int_{\Omega_l} \mathrm{card}(A_{\omega_l}, f) d\mu_l(\omega_l) \le n_1 + n_2, \end{aligned}$$

and, by (25) and (27),

$$\begin{aligned} e(S, A, F) &= \sup_{f \in F} \int_\Omega \|S(f) - A_\omega(f)\| d\mu(\omega) \\ &\le \sup_{f \in F} \sum_{l=1,2} \int_{\Omega_l} \|S_l(f) - A_{l,\omega_l}(f)\| d\mu_l(\omega_l) \\ &\le \sum_{l=1,2} e_{n_l}^{\mathrm{ran}}(S_l, F) + 2\delta. \end{aligned}$$

□

Finally, we study multiplicativity properties of the minimal error quantities. Here we suppose $K = \mathbb{K}$.

Proposition 3. *Let X be a normed space and assume $F \subseteq X$ and $\Lambda \subseteq X^\#$, where $X^\#$ denotes the algebraic dual, that is, the space of all linear (not necessarily continuous) functionals on X. Furthermore, let $J : F \to X$ be the embedding map, let $T : X \to G$ be a linear operator and assume that $S = TJ$. Then for all $n_1, n_2 \in \mathbb{N}_0$,*

$$e_{n_1+n_2}^{\mathrm{det}}(S, F) \le e_{n_1}^{\mathrm{det}}(J, F)\, e_{n_2}^{\mathrm{det}}(T, B_X), \tag{28}$$

and

$$e_{n_1+n_2}^{\mathrm{ran}}(S, F) \le e_{n_1}^{\mathrm{det}}(J, F)\, e_{n_2}^{\mathrm{ran}}(T, B_X). \tag{29}$$

Proof. Let

$$\begin{aligned} \mathcal{P}_1 &= (F, X, J, \mathbb{K}, \Lambda) \\ \mathcal{P}_2 &= (B_X, G, T, \mathbb{K}, \Lambda) \end{aligned}$$

(and, as before, $\mathcal{P} = (F, G, S, \mathbb{K}, \Lambda)$). To prove the deterministic case (28), let $\delta > 0$ and let $A_l \in \mathcal{A}^{\mathrm{det}}(\mathcal{P}_l)$ $(l = 1, 2)$ be deterministic algorithms

$$A_l = ((L_{l,i})_{i=1}^\infty, (\tau_{l,i})_{i=0}^\infty, (\varphi_{l,i})_{i=0}^\infty),$$

satisfying

$$\mathrm{card}(A_1, F) \le n_1, \quad \mathrm{card}(A_2, B_X) \le n_2, \tag{30}$$

$$e(J, A_1, F) = \sup_{f \in F} \|Jf - A_1(f)\| \le e_{n_1}^{\det}(J, F) + \delta := \theta. \tag{31}$$

$$e(T, A_2, B_X) = \sup_{g \in B_X} \|Tg - A_2(g)\| \le e_{n_2}^{\det}(T, B_X) + \delta. \tag{32}$$

We define an extended algorithm A in a way similar to the proof of Lemma 3. Put

$$Z_i = K^i \times \{0, \dots, i\} \times \{0, 1\}.$$

Set

$$z_0 = \begin{cases} (k^*, 0, 0) & \text{if} \quad \tau_{1,0}(k^*) = 0 \\ (k^*, 0, 1) & \text{if} \quad \tau_{1,0}(k^*) = 1. \end{cases}$$

For $z = (q, m, b) \in Z_i$, with $q = q_1 \oplus q_2$, $q_1 \in K^m$, $q_2 \in K^{i-m}$, define

$$\begin{aligned} L_{i+1}(z) &= \begin{cases} L_{1,i+1}(q) & \text{if} \quad b = 0 \\ L_{2,i+1-m}(q_2) & \text{if} \quad b = 1, \end{cases} \\ U_{i+1}(z, k) &= \begin{cases} (q \oplus k, m+1, 0) & \text{if} \quad b = 0 \quad \text{and} \quad \tau_{1,i+1}(q \oplus k) = 0 \\ (q \oplus k, m+1, 1) & \text{if} \quad b = 0 \quad \text{and} \quad \tau_{1,i+1}(q \oplus k) = 1 \\ (q \oplus \theta^{-1}(k - \varphi_{1,m}(q_1)(L_{i+1}(z))), m, 1) & \text{if} \quad b = 1 \end{cases} \end{aligned}$$

(concerning the last line: note that $\varphi_{1,m}(q_1) \in X$). Furthermore, let

$$\begin{aligned} \tau_i(z) &= \begin{cases} 0 & \text{if} \quad b = 0 \\ \tau_{2,i-m}(q_2) & \text{if} \quad b = 1, \end{cases} \\ \varphi_i(z) &= \begin{cases} 0 & \text{if} \quad b = 0 \\ T\varphi_{1,m}(q_1) + \theta\varphi_{2,i-m}(q_2) & \text{if} \quad b = 1. \end{cases} \end{aligned}$$

Let $f \in F$ and put

$$g = \theta^{-1}(Jf - A_1(f)) \tag{33}$$

It follows from (31) that

$$g \in B_X. \tag{34}$$

Let $(z_{1,i})_{i=0}^\infty$ be the computational sequence of A_1 at input f and $(z_{2,i})_{i=0}^\infty$ the computational sequence of A_2 at input g. Denote $\mathrm{card}(A_1, f) = \nu_1$, $\mathrm{card}(A_2, g) = \nu_2$, and $\lambda_{l,i} = L_{l,i}(z_{l,i-1})$ for $i \ge 1$ and $l = 1, 2$. From (30) and (34) we infer

$$\nu_1 \le n_1, \quad \nu_2 \le n_2. \tag{35}$$

Now we show by induction that the computational sequence $(z_i)_{i=0}^{\infty}$ of A at input f satisfies

$$z_i = \begin{cases} (z_{1,i}, i, 0) & \text{if} \quad i < \nu_1 \\ (z_{1,\nu_1} \oplus z_{2,i-\nu_1}, \nu_1, 1) & \text{if} \quad i \geq \nu_1. \end{cases} \tag{36}$$

The induction start and the induction step from i to $i+1$ for $i < \nu_1$ is identical to that in the proof of Lemma 3. We skip it here. Now suppose (36) holds for some $i \geq \nu_1$. We show it for $i+1$. By assumption, $z_i = (z_{1,\nu_1} \oplus z_{2,i-\nu_1}, \nu_1, 1)$. Consequently,

$$L_{i+1}(z_i) = L_{2,i+1-\nu_1}(z_{2,i-\nu_1}) = \lambda_{2,i+1-\nu_1}.$$

Moreover,

$$\begin{aligned} z_{i+1} &= U_{i+1}(z_i, f(L_{i+1}(z_i))) = U_{i+1}(z_i, f(\lambda_{2,i+1-\nu_1})) \\ &= (z_{1,\nu_1} \oplus z_{2,i-\nu_1} \oplus \theta^{-1}(f(\lambda_{2,i+1-\nu_1}) - \varphi_{1,\nu_1}(z_{1,\nu_1})(\lambda_{2,i+1-\nu_1})), \nu_1, 1) \\ &= (z_{1,\nu_1} \oplus z_{2,i-\nu_1} \oplus g(\lambda_{2,i+1-\nu_1}), \nu_1, 1) \\ &= (z_{1,\nu_1} \oplus z_{2,i+1-\nu_1}, \nu_1, 1). \end{aligned}$$

This proves (36). It follows that

$$\begin{aligned} \tau_i(z_i) &= 0 \quad \text{if} \quad i < \nu_1 + \nu_2 \\ \tau_i(z_i) &= 1 \quad \text{if} \quad i = \nu_1 + \nu_2. \end{aligned}$$

Therefore, we have

$$\operatorname{card}(A, f) = \nu_1 + \nu_2 = \operatorname{card}(A_1, f) + \operatorname{card}(A_2, g), \tag{37}$$

which, together with (35) implies

$$\operatorname{card}(A, F) \leq n_1 + n_2. \tag{38}$$

For $\nu = \operatorname{card}(A, f)$, we get

$$z_\nu = (z_{1,\nu_1} \oplus z_{2,\nu_2}, \nu_1, 1),$$

and thus,

$$\begin{aligned} A(f) &= \varphi_\nu(z_\nu) = T\varphi_{1,\nu_1}(z_{1,\nu_1}) + \theta\varphi_{2,\nu_2}(z_{2,\nu_2}) \\ &= TA_1(f) + \theta A_2(g). \end{aligned} \tag{39}$$

Consequently,

$$\begin{aligned}
&\|Sf - A(f)\| \\
&\quad = \|TJf - TA_1(f) - \theta A_2(g)\| \\
&\quad = \theta\|T(\theta^{-1}(Jf - A_1(f))) - A_2(g)\| \\
&\quad = \theta\|Tg - A_2(g)\| \qquad (40)\\
&\quad \le \theta e(T, A_2, B_X), \qquad (41)
\end{aligned}$$

where we used (34). According to Lemma 1, we can replace the extended algorithm A by an algorithm (keeping the notation A), so that (38) and (41) are preserved.

Together with (31), (32), (38), and (41) this gives

$$e_{n_1+n_2}^{\det}(S,F) \le e(S,A,F) \le (e_{n_1}^{\det}(J,F)+\delta)(e_{n_2}^{\det}(T,B_X)+\delta),$$

and the result for the deterministic case follows, since $\delta > 0$ was arbitrary.

Now we show (29). We choose $A_1 \in \mathcal{A}^{\det}(\mathcal{P}_1)$ and $A_2 \in \mathcal{A}^{\mathrm{ran}}(\mathcal{P}_2)$,

$$A_2 = ((\Omega, \Sigma, \mu), (A_{2,\omega})_{\omega\in\Omega})$$

so that (30) and (31) hold, and furthermore

$$e(T, A_2, B_X) = \sup_{g\in B_X} \int_\Omega \|Tg - A_{2,\omega}(g)\| d\mu(\omega) \le e_{n_2}^{\mathrm{ran}}(T, B_X) + \delta.$$

Then we define for each $\omega \in \Omega$ an algorithm $A_\omega \in \mathcal{A}^{\det}(\mathcal{P})$ as we did in the previous proof, just with A_2 replaced by $A_{2,\omega}$, and put

$$A = ((\Omega, \Sigma, \mu), (A_\omega)_{\omega\in\Omega}).$$

Let $f \in F$ and let g be given by (33). Then we get from (37)

$$\mathrm{card}(A_\omega, f) = \mathrm{card}(A_1, f) + \mathrm{card}(A_{2,\omega}, g),$$

and, by (39),

$$A_\omega(f) = TA_1(f) + \theta A_{2,\omega}(g).$$

Since $g \in B_X \subseteq \mathrm{Dom}(A_2)$, the measurability and separability requirements are satisfied, therefore $F \subseteq \mathrm{Dom}(A)$. It follows that

$$\mathrm{card}(A, F) \le n_1 + n_2.$$

For the error we get from (40),

$$\begin{aligned}\int_\Omega \|Sf - A_\omega(f)\| d\mu(\omega) &= \theta \int_\Omega \|Tg - A_{2,\omega}(g)\| d\mu(\omega) \\ &\le \theta e(T, A_2, B_X),\end{aligned}$$

and hence

$$e_{n_1+n_2}^{\mathrm{ran}}(S, F) \le e(S, A, F) \le (e_{n_1}^{\mathrm{det}}(J, F) + \delta)(e_{n_2}^{\mathrm{ran}}(T, B_X) + \delta),$$

which proves (29).

□

Remark. In relation (29) one would expect $e_{n_1}^{\mathrm{ran}}(J, F)$ in place of $e_{n_1}^{\mathrm{det}}(J, F)$. The proof above does not work in that case, since the randomized analogue of (31) does not give the crucial scaling (33) with (34). A way out is to require linearity of the algorithm for T (see [5], Proposition 3 for a result in that direction). Alternatively, one could pass to the probabilistic setting. Such an argument is, in fact, part of the proof of the quantum version of this proposition, see [8], Proposition 1. Here we do not pursue this topic any further since relation (29) is sufficient for our purposes.

4 Upper Bounds

This section contains the proof of the upper bounds in Theorem 1. A major ingredient of our analysis is a result of Krasovskij [12] on the Green's function of $\mathscr{L}$. To state it let us introduce the following class of kernels (compare also [9], where integral operators with such kernels are investigated).

Let $d, d_1 \in \mathbb{N}$, $d_1 \le d$, and let Q_1 be the closure of an open bounded set in $\mathbb{R}^{d_1}$, which we identify with a subset of $\mathbb{R}^d$ by identifying $\mathbb{R}^{d_1}$ with $\mathbb{R}^{d_1} \times \{0^{(d-d_1)}\}$. Let Q_2 be a bounded Lebesgue measurable subset of $\mathbb{R}^d$ of positive Lebesgue measure. Let $\mathrm{diag}(Q_1, Q_2) := \{(x, x) : x \in Q_1 \cap Q_2\}$.

Given $s \in \mathbb{N}$ and $\sigma \in \mathbb{R}$, $-d < \sigma < +\infty$ we denote by $\mathcal{C}^{s,\sigma}(Q_1, Q_2)$ the set of all Lebesgue measurable functions $k : Q_1 \times Q_2 \setminus \mathrm{diag}(Q_1, Q_2) \to \mathbb{C}$ with the property that there is a constant $c > 0$ such that for all $y \in Q_2$

1. $k(x, y)$ is s-times continuously differentiable with respect to x on $Q_1^0 \setminus \{y\}$, where Q_1^0 means the interior of Q_1, as a subset of $\mathbb{R}^{d_1}$,
2. for all multiindices $\alpha \in N_0^{d_1}$ with $0 \le |\alpha| = \alpha_1 + \cdots + \alpha_{d_1} \le s$ the α-th partial derivative of k with respect to the x-variables, which we

denote by $D_x^\alpha k(x,y)$, satisfies the estimate

$$|D_x^\alpha k(x,y)| \le c \begin{cases} |x-y|^{\sigma-|\alpha|}+1 & \text{if} \quad \sigma-|\alpha| \neq 0 \\ |\ln|x-y||+1 & \text{if} \quad \sigma-|\alpha| = 0 \end{cases} \tag{42}$$

for all $x \in Q_1^0 \setminus \{y\}$, and

3. for all $\alpha \in N_0^{d_1}$ with $0 \le |\alpha| \le s$ the functions $D_x^\alpha k(x,y)$ have continuous extensions to $Q_1 \setminus \{y\}$.

We also need to consider the case $d_1 = 0$. Here we put $Q_1 = \{x_0\}$, where x_0 is any point of $\mathbb{R}^d$. The set $\mathcal{C}^{s,\sigma}(Q_1,Q_2)$ does not depend on s and consists of all functions $k(x_0,y)$ which are Lebesgue measurable in y and satisfy

$$|k(x_0,y)| \le c \begin{cases} |x_0-y|^{\sigma}+1 & \text{if} \quad \sigma \neq 0 \\ |\ln|x_0-y||+1 & \text{if} \quad \sigma = 0 \end{cases} \qquad (y \in Q_2 \setminus \{x_0\}) \tag{43}$$

for a certain $c > 0$. For $k \in \mathcal{C}^{s,\sigma}(Q_1,Q_2)$ let $\|k\|_{\mathcal{C}^{s,\sigma}}$ denote the smallest $c > 0$ satisfying (42) or (43), respectively. It is readily verified that $\|\,.\,\|_{\mathcal{C}^{s,\sigma}}$ is a norm, which turns $\mathcal{C}^{s,\sigma}(Q_1,Q_2)$ into a Banach space. For $k \in \mathcal{C}^{s,\sigma}(Q_1,Q_2)$ we denote by T_k the integral operator

$$(T_k f)(x) = \int_{Q_2} k(x,y) f(y) dy \quad (x \in Q_1)$$

acting from $L_\infty(Q_2)$ to $L_\infty(Q_1)$ (T_k will also be considered as acting in various other suitable function spaces, which will be clear from the context or will be mentioned explicitly). Let us furthermore denote

$$\mathcal{C}^{\infty,\sigma}(Q_1,Q_2) := \bigcap_{s\in\mathbb{N}} \mathcal{C}^{s,\sigma}(Q_1,Q_2).$$

Now let M, Q and S be as defined in section 2. By Krasovskij [12], Theorem 3.3 and Corollary, there is a kernel $k \in \mathcal{C}^{\infty,2m-d}(Q,Q)$ such that for all $f \in C^{\kappa_0}(Q)$ the solution u of (8), (9) satisfies

$$u(x) = \int_Q k(x,y) f(y) dy \quad (x \in Q). \tag{44}$$

Thus, we have, in particular,

$$(Sf)(x) = (T_k f)(x) \quad (x \in M),$$

that means, $S = T_k$, with $T_k : C^r(Q) \to L_\infty(M)$, and we have to investigate the approximation of $T_k f$.

First we consider the case of simple domains $Q_1 = [0,1]^{d_1}$ and $Q_2 = [0,1]^d$, where $d_1 \in \mathbb{N}_0$, $d \in \mathbb{N}$, $d_1 \le d$. We study the approximation of $T_k f$ with $k \in \mathcal{C}^{s,\sigma}(Q_1, Q_2)$ a fixed kernel, $f \in C^r(Q_2)$, and the operator T_k is considered as acting from $C^r(Q_2)$ to $L_\infty(Q_1)$, so we let $F = B_{C^r(Q_2)}$, $G = L_\infty(Q_1)$, and $\Lambda = \{\delta_x : x \in Q_2\}$ (for $d_1 = 0$ we consider $Q_1 = \{0\}$ and the space $L_\infty(Q_1)$ is replaced by $\mathbb{C}$).

Proposition 4. *Assume* $0 \le d_1 \le d$, $s \in \mathbb{N}$, $s > \frac{d_1}{2}$, $\sigma \in \mathbb{R}$, $-d < \sigma < +\infty$, $r \in \mathbb{N}$. *Then there is a constant* $c > 0$ *such that for all* $k \in B_{\mathcal{C}^{s,\sigma}(Q_1,Q_2)}$ *and* $n \in \mathbb{N}$ *with* $n \ge 2$,

$$e_n^{\mathrm{ran}}(T_k, B_{C^r(Q_2)}) \le c n^{-\min\left(\frac{r+d+\sigma}{d_1}, \frac{r}{d}+\frac{1}{2}\right)} (\log n)^{\kappa(\sigma)}.$$

where $\kappa(\sigma)$ *is as defined in (10).*

Proof. First we assume $\frac{d+\sigma}{d_1} \ge \frac{1}{2}$ or $d_1 = d$. We represent $T_k = S_1 J$, where J is the identical embedding $C^r(Q_2) \to C(Q_2)$, and S_1 is T_k, considered as an operator from $C(Q_2)$ to $L_\infty(Q_1)$. It is well-known that

$$e_n^{\det}(J, B_{C^r(Q_2)}) \le c n^{-\frac{r}{d}}.$$

By [9], Theorem 1,

$$e_n^{\mathrm{ran}}(S_1, B_{C(Q_2)}) \le c n^{-\min\left(\frac{d+\sigma}{d_1}, \frac{1}{2}\right)} (\log n)^{\kappa(\sigma)}$$

(the constants in this proof do not depend on k). From Proposition 3,

$$e_{2n}^{\mathrm{ran}}(T_k, B_{C^r(Q_2)}) \le e_n^{\mathrm{ran}}(S_1, B_{C(Q_2)}) e_n^{\det}(J, B_{C^r(Q_2)}),$$

and the desired result follows.

Next we assume $\frac{d+\sigma}{d_1} < \frac{1}{2}$ (which implies $d_1 \ne 0$) and $d_1 < d$. Let

$$m = \left\lceil \frac{\log n}{d_1} \right\rceil, \tag{45}$$

$$H_l = [0,1]^{d_1} \times \left(2^{-l}[0,1]^{d-d_1} \setminus 2^{-(l+1)}[0,1]^{d-d_1}\right) \quad (l = 0, \dots, m-1)$$

and

$$H_m = [0,1]^{d_1} \times \left(2^{-m}[0,1]^{d-d_1}\right).$$

Clearly, $Q_2 = \bigcup_{l=0}^m H_l$. Put

$$p_l = \left\lceil 2^{\left(\frac{d_1}{d}-\delta_1\right)(m-l)-\delta_2 l} \right\rceil \quad (0 \le l \le m), \tag{46}$$

where $\delta_1, \delta_2 \geq 0$ will be fixed later. Note that $p_m = 1$. Let $H_l = \bigcup_{i=1}^{n_l} H_{li}$ be the partition of H_l into subcubes of mutually disjoint interior and of sidelength

$$\varepsilon_l = \begin{cases} 2^{-(l+1)} p_l^{-1} & \text{if} \quad 0 \leq l < m \\ 2^{-m} & \text{if} \quad l = m. \end{cases}$$

The number of such cubes is

$$n_l = \begin{cases} (2^{d-d_1} - 1)\, 2^{d_1(l+1)} p_l^d = c\, 2^{d_1 l} p_l^d & \text{if} \quad 0 \leq l < m \\ 2^{d_1 m} & \text{if} \quad l = m. \end{cases} \tag{47}$$

Since $1 \leq d_1 < d$ and $m \geq 1$, it follows that $n_l \geq 2$ for $0 \leq l \leq m$. Now let P_l be the composition of tensor product Lagrange interpolation on H_{li} of degree $\max(r-1, 1)$ (compare [9], section 2). Then for $0 \leq l \leq m$,

$$\|f - P_l f\|_{C(H_l)} \leq c\varepsilon_l^r \leq c\, 2^{-rl} p_l^{-r} \|f\|_{C^r(H_l)}. \tag{48}$$

Let $J_l : C^r(H_l) \to C(H_l)$ be the embedding operator. It follows that

$$e^{\mathrm{det}}_{c_1 n_l}(J_l, B_{C^r(H_l)}) \leq c\, 2^{-rl} p_l^{-r}, \tag{49}$$

where c_1 is the number of interpolation nodes in each subcube H_{li} (this number depends only on r and d). Define

$$k_l(x, y) = k(x, y)\chi_{H_l}(y) \quad (x \in Q_1,\, y \in Q_2).$$

Then $k_l \in \mathcal{C}^{s,\sigma}(Q_1, Q_2)$ and

$$\|k_l\|_{\mathcal{C}^{s,\sigma}(Q_1,Q_2)} \leq \|k\|_{\mathcal{C}^{s,\sigma}(Q_1,Q_2)}. \tag{50}$$

We have

$$T_k = \sum_{l=0}^{m} T_{k_l} E_l J_l R_l, \tag{51}$$

where $R_l : C^r(Q_2) \to C^r(H_l)$ is the restriction operator, $E_l : C(H_l) \to \mathcal{L}_\infty(Q_2)$ is the operator of extension by zero, and T_{k_l} is considered as an operator from $\mathcal{L}_\infty(Q_2)$ to $L_\infty(Q_1)$. Here $\mathcal{L}_\infty(Q_2)$ is the linear space of all Lebesgue measurable essentially bounded real-valued functions on Q_2, equipped with the seminorm

$$|f|_{\mathcal{L}_\infty(Q_2)} = \operatorname{ess\,sup}_{x \in Q_2} |f(x)|.$$

The space $\mathcal{L}_\infty(Q_2)$ consists of functions defined everywhere on Q_2. In contrast, the normed space $L_\infty(Q_1)$, which appears as the target space of T_k, consists of equivalence classes. The reason for this choice is that in $\mathcal{L}_\infty(Q_2)$ function values are defined (while they are not in $L_\infty(Q_2)$). Define

$$\mathcal{B}_{\mathcal{L}_\infty(Q_2)} = \{f \in \mathcal{L}_\infty(Q_2) \,:\, |f|_{\mathcal{L}_\infty(Q_2)} \le 1\}.$$

Put $\sigma_1 = d_1/2 - d$, so that

$$\frac{d+\sigma_1}{d_1} = \frac{1}{2}.$$

By assumption, $\frac{d+\sigma}{d_1} < \frac{1}{2}$, therefore $\sigma_1 > \sigma$ and $\sigma < 0$. Let Q_1^0 be the d_1-dimensional interior of Q_1, that is, $Q_1^0 = (0,1)^{d_1} \times \{0^{(d-d_1)}\}$. It follows from (50) that for $0 \le l < m$, $|\alpha| \le s$, $y \in Q_2$, $x \in Q_1^0 \setminus \{y\}$ we have

$$|D_x^\alpha k_l(x,y)| \le c|x-y|^{\sigma-|\alpha|} \le c\, 2^{(\sigma_1-\sigma)(l+1)}|x-y|^{\sigma_1-|\alpha|},$$

since, by definition, $k_l(x,y) = 0$ whenever $|x-y| < 2^{-(l+1)}$. Consequently,

$$\|k_l\|_{\mathcal{C}^{s,\sigma_1}(Q_1,Q_2)} \le c 2^{(\sigma_1-\sigma)l} \quad (0 \le l < m). \tag{52}$$

Next we show that

$$e_{n_l}^{\mathrm{ran}}(T_{k_l}, \mathcal{B}_{\mathcal{L}_\infty(Q_2)}) \le c 2^{-(d+\sigma)l} p_l^{-\frac{d}{2}} (\log n_l)^{\frac{3}{2}} \quad (0 \le l \le m). \tag{53}$$

Indeed, for $0 \le l < m$ we conclude from (52) and Theorem 1 of [9] that

$$\begin{aligned} e_{n_l}^{\mathrm{ran}}(T_{k_l}, \mathcal{B}_{\mathcal{L}_\infty(Q_2)}) &\le c n_l^{-\frac{1}{2}} 2^{(\sigma_1-\sigma)l} (\log n_l)^{\frac{3}{2}} \\ &\le c 2^{-\frac{d_1 l}{2}} p_l^{-\frac{d}{2}} 2^{(\frac{d_1}{2}-d-\sigma)l} (\log n_l)^{\frac{3}{2}} \\ &= c 2^{-(d+\sigma)l} p_l^{-\frac{d}{2}} (\log n_l)^{\frac{3}{2}}. \end{aligned}$$

For $l = m$ we have by (50) and Theorem 1 of [9],

$$e_{n_m}^{\mathrm{ran}}(T_{k_m}, \mathcal{B}_{\mathcal{L}_\infty(Q_2)}) \le c n_m^{-\frac{d+\sigma}{d_1}} (\log n_m)^{\frac{d+\sigma}{d_1}} \le c 2^{-(d+\sigma)m} (\log n_m)^{\frac{d+\sigma}{d_1}},$$

which implies (53) also in this case, since $p_m = 1$ and $\frac{d+\sigma}{d_1} < \frac{1}{2}$. Setting

$$\bar{n} = (c_1+1) \sum_{l=0}^{m} n_l$$

with c_1 from (49), we conclude from (51) and Proposition 2,

$$e_{\bar{n}}^{\mathrm{ran}}(T_k, B_{C^r(Q_2)}) \le \sum_{l=0}^{m} e_{(c_1+1)n_l}^{\mathrm{ran}}(T_{k_l} E_l J_l R_l, B_{C^r(Q_2)}). \tag{54}$$

The mapping R_l is of the form (14) with $\kappa = 1$, $\Lambda = \{\delta_x : x \in Q_2\}$, and $\widetilde{\Lambda} = \{\delta_x : x \in H_l\}$, and satisfies $R_l(B_{C^r(Q_2)}) \subseteq B_{C^r(H_l)}$, hence, by Proposition 1,

$$e^{\mathrm{ran}}_{(c_1+1)n_l}(T_{k_l}E_lJ_lR_l, B_{C^r(Q_2)}) \le e^{\mathrm{ran}}_{(c_1+1)n_l}(T_{k_l}E_lJ_l, B_{C^r(H_l)}). \qquad (55)$$

Furthermore, by Proposition 3,

$$e^{\mathrm{ran}}_{(c_1+1)n_l}(T_{k_l}E_lJ_l, B_{C^r(H_l)}) \le e^{\mathrm{ran}}_{n_l}(T_{k_l}E_l, B_{C(H_l)})e^{\mathrm{det}}_{c_1n_l}(J_l, B_{C^r(H_l)}). \qquad (56)$$

The mapping E_l is also of the form (14) with $\kappa = 1$, $\Lambda = \{\delta_x : x \in H_l\}$, and $\widetilde{\Lambda} = \{\delta_x : x \in Q_2\}$, and $E_l(B_{C(H_l)}) \subseteq \mathcal{B}_{\mathcal{L}_\infty(Q_2)}$. Consequently,

$$e^{\mathrm{ran}}_{n_l}(T_{k_l}E_l, B_{C(H_l)}) \le e^{\mathrm{ran}}_{n_l}(T_{k_l}, \mathcal{B}_{\mathcal{L}_\infty(Q_2)}). \qquad (57)$$

Joining (55), (56), and (57), we get

$$e^{\mathrm{ran}}_{(c_1+1)n_l}(T_{k_l}E_lJ_lR_l, B_{C^r(Q_2)}) \le e^{\mathrm{ran}}_{n_l}(T_{k_l}, \mathcal{B}_{\mathcal{L}_\infty(Q_2)})e^{\mathrm{det}}_{c_1n_l}(J_l, B_{C^r(H_l)}).$$

Together with (54), (49), (53), and (46) we obtain

$$\begin{aligned} & e^{\mathrm{ran}}_{\bar{n}}(T_k, B_{C^r(Q_2)}) && (58)\\ \le\ & \sum_{l=0}^{m} e^{\mathrm{ran}}_{n_l}(T_{k_l}, \mathcal{B}_{\mathcal{L}_\infty(Q_2)})e^{\mathrm{det}}_{c_1n_l}(J_l, B_{C^r(H_l)}) \\ \le\ & c\sum_{l=0}^{m} 2^{-(r+d+\sigma)l} p_l^{-\left(r+\frac{d}{2}\right)} (\log n_l)^{\frac{3}{2}} \\ \le\ & cm^{\frac{3}{2}} \sum_{l=0}^{m} 2^{-\left(r+d+\sigma-\delta_2\left(r+\frac{d}{2}\right)\right)l-\left(r+\frac{d}{2}\right)\left(\frac{d_1}{d}-\delta_1\right)(m-l)}. && (59)\end{aligned}$$

Moreover, using (47), (46), and (45), we get

$$\begin{aligned}\bar{n} &= (c_1+1)\sum_{l=0}^{m} n_l \le c\sum_{l=0}^{m} 2^{d_1 l}\left(2^{d_1(m-l)-\delta_1 d(m-l)-\delta_2 dl}+1\right) \\ &\le \begin{cases} c2^{d_1 m} \le cn & \text{if } \delta_1 > 0 \text{ or } \delta_2 > 0 \\ cm2^{d_1 m} \le cn\log n & \text{if } \delta_1 = \delta_2 = 0. \end{cases} \qquad (60)\end{aligned}$$

Now we derive the final estimates. First we consider the case $r + d + \sigma > (r + \frac{d}{2})\frac{d_1}{d}$, that is, $\frac{r+d+\sigma}{d_1} > \frac{r}{d} + \frac{1}{2}$. Here we choose $\delta_1 = 0$ and $\delta_2 > 0$ so small that we still have

$$r + d + \sigma - \delta_2\left(r + \frac{d}{2}\right) > \left(r + \frac{d}{2}\right)\frac{d_1}{d}.$$

Hence, from (59) and (45),

$$\begin{aligned} e_{\bar n}^{\mathrm{ran}}(T_k, B_{C^r(Q_2)}) &\le cm^{\frac32} 2^{-\left(r+\frac d2\right)\frac{d_1}{d}m} = cm^{\frac32}2^{-\left(\frac rd+\frac12\right)d_1 m} \\ &\le cn^{-\left(\frac rd+\frac12\right)}(\log n)^{\frac32}. \end{aligned}$$

This together with (60), the monotonicity of the numbers e_n^{ran} in n, and a suitable scaling gives

$$e_n^{\mathrm{ran}}(T_k, B_{C^r(Q_2)}) \le cn^{-\left(\frac rd+\frac12\right)}(\log n)^{\frac32}.$$

Next we assume $r+d+\sigma = (r+\frac d2)\frac{d_1}{d}$ and put $\delta_1=\delta_2=0$. By (59),

$$\begin{aligned} e_{\bar n}^{\mathrm{ran}}(T_k, B_{C^r(Q_2)}) &\le cm^{\frac52} 2^{-\left(r+\frac d2\right)\frac{d_1}{d}m} = cm^{\frac52}2^{-\left(\frac rd+\frac12\right)d_1 m} \\ &\le cn^{-\left(\frac rd+\frac12\right)}(\log n)^{\frac52}. \end{aligned}$$

By (60) and scaling,

$$e_n^{\mathrm{ran}}(T_k, B_{C^r(Q_2)}) \le cn^{-\left(\frac rd+\frac12\right)}(\log n)^{\frac rd+3}.$$

Finally, if $r+d+\sigma < (r+\frac d2)\frac{d_1}{d}$, we set $\delta_2=0$ and choose $\delta_1>0$ so that

$$r+d+\sigma < \left(r+\frac d2\right)\left(\frac{d_1}{d}-\delta_1\right).$$

From (59),

$$e_{\bar n}^{\mathrm{ran}}(T_k, B_{C^r(Q_2)}) \le cm^{\frac32}2^{-(r+d+\sigma)m} \le cn^{-\frac{r+d+\sigma}{d_1}}(\log n)^{\frac32}.$$

which implies

$$e_n^{\mathrm{ran}}(T_k, B_{C^r(Q_2)}) \le cn^{-\frac{r+d+\sigma}{d_1}}(\log n)^{\frac32}$$

and completes the proof. □

Proof of the upper bound in Theorem 1. We prove a slightly stronger statement: We show that the upper bound holds even for the smaller sets of information functionals $\Lambda = \{\delta_x : x \in Q\}$. Let M and Q be as defined in section 2. Let $k \in \mathcal{C}^{\infty,2m-d}(Q,Q)$ be such that (44) holds. Let U_x, Φ_x $(x\in M)$ be as required for M being a C^∞ submanifold of Q. We let

$$W'_x = \begin{cases} W'_+ & \text{if} \quad x\in M\cap\partial Q \\ W' & \text{if} \quad x \in M\cap Q^0. \end{cases}$$

Then we have $\Phi_x(W_x') = U_x \cap Q$. Furthermore, we set

$$V_x' = \begin{cases} [0,1/2] \times [-1/2,1/2]^{d-1} & \text{if} \quad x \in M \cap \partial Q \\ [-1/2,1/2]^d & \text{if} \quad x \in M \cap Q^0. \end{cases}$$

Put $V_x = \Phi_x(V_x')$. Clearly $M \subseteq \bigcup_{x\in M} V_x$, and since the set M is compact, we can choose a finite set $\{x_1, \dots, x_p\}$ such that $V_{x_1}, \dots, V_{x_p}$ cover M. For simplicity of notation we replace the subscript x_i by i, thus writing $\Phi_i, U_i, V_i, V_i', W_i'$. Set $M_i = M \cap V_i$ and $M_i' = \Phi_i^{-1}(M_i)$. Then

$$M_i' = \{y \in V_i' \,:\, y_{d_1+1} = \dots = y_d = 0\}. \tag{61}$$

Let

$$C_i = M_i \setminus \bigcup_{j<i} M_j$$

and denote the operator from $L_\infty(C_i)$ to $L_\infty(M)$ of extension by zero on $M \setminus C_i$ by E_i. Now fix $1 \le i \le p$. For $f \in C^r(Q)$ and $x \in C_i$ we have

$$(T_k f)(x) = \int_{U_i\cap Q} k(x,y)f(y)dy + \int_{Q\setminus U_i} k(x,y)f(y)dy. \tag{62}$$

The first summand can be transformed as follows: Let $x' = \Phi_i^{-1}(x)$.

$$\begin{aligned} \int_{U_i\cap Q} k(x,y)f(y)dy &= \int_{W_i'} k(\Phi_i(x'),\Phi_i(y'))f(\Phi_i(y'))|\det(\mathcal{J}_{\Phi_i})(y')|dy' \\ &= \int_{W_i'} k_i'(x',y')f_i'(y')dy' = (T_{k_i'}f_i')(x') \\ &= (T_{k_i'}f_i')(\Phi_i^{-1}(x)) = (X_i T_{k_i'} Y_i f)(x), \end{aligned} \tag{63}$$

where we defined for $z' \in M_i'$, $y' \in W_i'$,

$$\begin{aligned} k_i'(z',y') &= k(\Phi_i(z'),\Phi_i(y'))|\det(\mathcal{J}_{\Phi_i}(y'))|, \\ f_i'(y') &= f(\Phi_i(y')), \end{aligned}$$

furthermore, $X_i : L_\infty(M_i') \to L_\infty(C_i)$ for $g \in L_\infty(M_i')$ by

$$(X_i g)(z) = g(\Phi_i^{-1}(z)) \quad (z \in C_i),$$

$Y_i : C^r(Q) \to C^r(W_i')$ for $f \in C^r(Q)$ by

$$(Y_i f)(z') = f(\Phi_i(z')) \quad (z' \in W_i'),$$

and the integral operator $T_{k_i'}$ is considered as acting from $C^r(W_i')$ to $L_\infty(M_i')$. Observe that

$$k_i' \in \mathcal{C}^{\infty,2m-d}(M_i', W_i'), \tag{64}$$

X_i and Y_i are bounded linear operators, and $\|X_i\| \le 1$.

Now we consider the second term of (62), for $f \in C^r(Q)$ and $x \in C_i$.

$$\begin{aligned} \int_{Q\setminus U_i} k(x,y)f(y)dy &= \int_{Q\setminus U_i} k(\Phi_i(x'),y)f(y)dy \\ &= \int_{Q\setminus U_i} k_i''(x',y)f(y)dy \\ &= (T_{k_i''}Z_i f)(x') = (X_i T_{k_i''} Z_i J f)(x). \end{aligned} \tag{65}$$

Here

$$k_i''(z',y) = k(\Phi_i(z'),y) \quad (z' \in M_i',\ y \in Q\setminus U_i),$$

$J : C^r(Q) \to C(Q)$ is the identical embedding, $Z_i : C(Q) \to C(Q\setminus U_i)$ the operator of restriction from Q to $Q\setminus U_i$, the operator $T_{k_i''}$ is considered as a mapping from $C(Q\setminus U_i)$ to $L_\infty(M_i')$, and X_i was defined above. We have

$$k_i'' \in \bigcap_{\sigma>0} \mathcal{C}^{\infty,\sigma}(M_i', Q\setminus U_i), \tag{66}$$

since by construction, $M_i = \Phi_i(M_i')$ is closed and contained in the interior of U_i. From (62), (63), and (65) we get for $f \in C^r(Q)$ and $x \in C_i$,

$$(T_k f)(x) = (X_i T_{k_i'} Y_i f)(x) + (X_i T_{k_i''} Z_i J f)(x),$$

hence we obtained the following representation

$$T_k = \sum_{i=1}^p E_i X_i T_{k_i'} Y_i + E_i X_i T_{k_i''} Z_i J. \tag{67}$$

Therefore, by Proposition 2,

$$\begin{aligned} & e_{3pn}^{\mathrm{ran}}(T_k, B_{C^r(Q)}) \\ & \le \sum_{i=1}^p e_{3n}^{\mathrm{ran}}(E_i X_i T_{k_i'} Y_i + E_i X_i T_{k_i''} Z_i J, B_{C^r(Q)}) \\ & \le \sum_{i=1}^p \left(e_n^{\mathrm{ran}}(E_i X_i T_{k_i'} Y_i, B_{C^r(Q)}) + e_{2n}^{\mathrm{ran}}(E_i X_i T_{k_i''} Z_i J, B_{C^r(Q)}) \right). \end{aligned}$$

Since Y_i is of the form (14), with $\kappa = 1$, $\Lambda = \{\delta_x : x \in Q\}$, and $\widetilde{\Lambda} = \{\delta_{x'} : x' \in W_i'\}$, moreover, $Y_i(B_{C^r(Q)}) \subseteq \|Y_i\| B_{C^r(W_i')}$ and $\|E_i X_i\| \le 1$, we have by Proposition 1,

$$\begin{aligned} e_n^{\mathrm{ran}}(E_i X_i T_{k_i'} Y_i, B_{C^r(Q)}) &\le e_n^{\mathrm{ran}}(T_{k_i'}, \|Y_i\| B_{C^r(W_i')}) \\ &= \|Y_i\| e_n^{\mathrm{ran}}(T_{k_i'}, B_{C^r(W_i')}). \end{aligned}$$

Furthermore, by Proposition 3,

$$\begin{aligned} & e_{2n}^{\mathrm{ran}}(E_i X_i T_{k_i''} Z_i J, B_{C^r(Q)}) \\ & \le e_n^{\mathrm{ran}}(E_i X_i T_{k_i''} Z_i, B_{C(Q)}) e_n^{\mathrm{det}}(J, B_{C^r(Q)}). \end{aligned}$$

Moreover, Z_i also has the form (14), with $\kappa = 1$, $\Lambda = \{\delta_x : x \in Q\}$, and $\widetilde{\Lambda} = \{\delta_x : x \in Q \setminus U_i\}$, and $Z_i(B_{C(Q)}) \subseteq B_{C(Q \setminus U_i)}$, so Proposition 1 gives

$$e_n^{\mathrm{ran}}(E_i X_i T_{k_i''} Z_i, B_{C(Q)}) \le e_n^{\mathrm{ran}}(T_{k_i''}, B_{C(Q \setminus U_i)}).$$

Thus we obtain

$$\begin{aligned} & e_{3pn}^{\mathrm{ran}}(T_k, B_{C^r(Q)}) \\ & \le c \sum_{i=1}^{p} \Big(e_n^{\mathrm{ran}}(T_{k_i'}, B_{C^r(W_i')}) \\ & \quad + e_n^{\mathrm{ran}}(T_{k_i''}, B_{C(Q \setminus U_i)}) e_n^{\mathrm{det}}(J, B_{C^r(Q)}) \Big). \end{aligned} \tag{68}$$

It is easily seen that $T_{k_i'}$ can be split into the sum of 2^{d-d_1} operators of the required form for Proposition 4 (just split W_i' into respective pieces and apply a scaling). Hence we conclude from (64) and Proposition 4 that

$$e_n^{\mathrm{ran}}(T_{k_i'}, B_{C^r(W_i')}) \le n^{-\min\left(\frac{r+2m}{d_1}, \frac{r}{d} + \frac{1}{2}\right)} (\log n)^{\kappa(2m-d)}, \tag{69}$$

where $\kappa(2m-d)$ is defined in (10). Moreover,

$$e_n^{\mathrm{det}}(J, B_{C^r(Q)}) \le c n^{-\frac{r}{d}}, \tag{70}$$

see, e.g., [20]. Furthermore, from (66) and Theorem 1 of [9],

$$e_n^{\mathrm{ran}}(T_{k_i''}, B_{C(Q \setminus U_i)}) \le c n^{-\frac{1}{2}}. \tag{71}$$

Relations (68)–(71) finally give

$$e_{3pn}^{\mathrm{ran}}(T_k, B_{C^r(Q)}) \le c n^{-\min\left(\frac{r+2m}{d_1}, \frac{r}{d} + \frac{1}{2}\right)} (\log n)^{\kappa(2m-d)},$$

and, since p is a constant, rescaling gives the desired result. □

5 Lower Bounds

In this section we prove the lower bounds in Theorem 1. Let again M and Q be as defined in section 2 and $k \in \mathcal{C}^{\infty,2m-d}(Q,Q)$ be such that (44) holds. We have $M \cap Q^0 \neq \emptyset$. Indeed, for $d_1 \geq 1$ this follows from the definition of M, while for $d_1 = 0$ we assumed that M consists of a single point, which is an inner point of Q. So let $x_0 \in M \cap Q^0$. Choose any function $u_0 \in C^\infty(Q)$ with all derivatives vanishing on ∂Q and $u_0(x_0) \neq 0$, and set $f_0 = \mathscr{L} u_0$. Then $f_0 \in C^r(Q)$ and $(Sf_0)(x_0) \neq 0$. Hence, there must be a $y_0 \in Q^0$, $y_0 \neq x_0$ such that $k(x_0, y_0) \neq 0$. Since $k(x,y)$ is infinitely differentiable in both variables for $x \neq y$ (see [12], Theorem 3.3 and Corollary), there is a closed neighborhood $U_0 \subset Q$ of x_0, a cube $V \subset Q$ with $U_0 \cap V = \emptyset$ and a $\vartheta \neq 0$ such that $\mathrm{Re}(\vartheta k(x,y)) \geq 1$ for $x \in U_0,\ y \in V$. Define

$$h(y) = \begin{cases} \left(\int_{M\cap U_0} k(x,y)\,dx\right)^{-1} & \text{if} \quad d_1 \geq 1 \\ k(x_0,y)^{-1} & \text{if} \quad d_1 = 0 \end{cases} \qquad (y \in V),$$

where the integral is taken with respect to the surface measure of M. Note that $h(y)$ is an infinitely differentiable function on V. Let $C_0^r(V)$ be the subspace of $C^r(V)$ consisting of those functions whose partial derivatives up to the order r vanish on ∂V. Define $X_0 : C_0^r(V) \to C^r(Q)$ by

$$(X_0 f)(y) = \begin{cases} h(y)f(y) & \text{if} \quad y \in V \\ 0 & \text{if} \quad y \notin V, \end{cases}$$

furthermore, $Y_0 : L_\infty(M) \to \mathbb{C}$ by

$$Y_0 g = \int_{M\cap U_0} g(x)\,dx$$

(if $d_1 = 0$, we replace $L_\infty(M)$ by $\mathbb{C}$, and let Y_0 be the identity), and $S_1 : C_0^r(V) \to \mathbb{C}$ as

$$S_1 f = \int_V f(y)\,dy.$$

Then we have

$$\begin{aligned} Y_0 S X_0 f &= \int_{M\cap U_0}\int_V k(x,y)h(y)f(y)\,dydx \\ &= \int_V f(y)h(y)\int_{M\cap U_0} k(x,y)\,dxdy \\ &= \int_V f(y)\,dy = S_1 f \end{aligned} \tag{72}$$

(with the obvious modifications for $d_1 = 0$). Moreover, X_0 and Y_0 are bounded linear operators, thus, in particular, $X_0(B_{C_0^r(V)}) \subseteq \|X_0\| B_{C^r(Q)}$, and X_0 is of the form (14) with $\Lambda = \{\delta_x^\alpha : x \in V^0, |\alpha| \le r\}$, $\widetilde{\Lambda} = \{\delta_x^\alpha : x \in Q, |\alpha| \le r\}$, and κ is the number of multiindices $\alpha \in \mathbb{N}_0^d$ with $|a| \le r$. Consequently, by Proposition 1 and (72), with $\varpi \in \{\mathrm{det}, \mathrm{ran}\}$,

$$\begin{aligned} e_{\kappa n}^{\varpi}(S_1, B_{C_0^r(V)}) &\le \|Y_0\| e_n^{\varpi}(S, \|X_0\| B_{C^r(Q)}) \\ &= \|X_0\| \|Y_0\| e_n^{\varpi}(S, B_{C^r(Q)}). \end{aligned}$$

It is well-known that

$$\begin{aligned} e_{\kappa n}^{\mathrm{det}}(S_1, B_{C_0^r(V)}) &\ge c n^{-\frac{r}{d}} \\ e_{\kappa n}^{\mathrm{ran}}(S_1, B_{C_0^r(V)}) &\ge c n^{-\frac{r}{d}-\frac{1}{2}}. \end{aligned}$$

Thus we conclude

$$\begin{aligned} e_n^{\mathrm{det}}(S, B_{C^r(Q)})) &\ge c n^{-\frac{r}{d}} \\ e_n^{\mathrm{ran}}(S, B_{C^r(Q)}) &\ge c n^{-\frac{r}{d}-\frac{1}{2}}. \end{aligned}$$

This proves the lower bound in the deterministic setting and in the randomized setting for the case $\frac{r}{d} + \frac{1}{2} \le \frac{r+2m}{d_1}$ (including the case $d_1 = 0$).

Now we assume $d_1 \ge 1$ and present another reduction, which will prove the remaining part. Let again $x_0 \in M \cap Q^0$ and let $U = U_{x_0}$ and $\Phi = \Phi_{x_0}$ be as described in the definition of a C^∞ submanifold. Let $M_0 = U \cap M$. So Φ^{-1} takes U to $W' = [-1,1]^d$ and M_0 to

$$\{y \in W' : y_{d_1+1} = \cdots = y_d = 0\} = M_0'.$$

We identify M_0' with $[-1,1]^{d_1}$. Let $C_0^{r+2m}(M_0')$ be the space of functions in $C^{r+2m}(M_0')$ which vanish together with all partial derivatives up to order $r+2m$ on $\partial M_0'$ (the boundary of M_0', considered as a subset of $\mathbb{R}^{d_1}$). Define $C_0^{r+2m}(W')$ and $C_0^{r+2m}(Q)$ in the same way.

We define an operator $E : C_0^{r+2m}(M_0') \to C_0^{r+2m}(W')$ as follows. If $d_1 < d$, we choose a C^∞ function ψ with support in $(-1,1)^{d-d_1}$ and $\psi(0) = 1$. For $g \in C_0^{r+2m}(M_0')$, we let

$$(Eg)(x) = g(x_1)\psi(x_2), \tag{73}$$

where $x = (x_1, x_2) \in W'$, $x_1 \in [-1,1]^{d_1}$, $x_2 \in [-1,1]^{d-d_1}$. If $d_1 = d$, then $M_0' = W'$, and we let E be the identity. Furthermore, we define the operator $X : C_0^{r+2m}(W') \to C_0^{r+2m}(Q)$ by

$$(Xf)(x) = \begin{cases} f(\Phi^{-1}(x)) & \text{if} \quad x \in U \\ 0 & \text{otherwise} \end{cases} \qquad \left(f \in C_0^{r+2m}(W'),\ x \in Q\right),$$

and $Y : L_\infty(M) \to L_\infty(M_0')$ by

$$(Yg)(y) = g(\Phi(y)) \qquad \big(g \in L_\infty(M),\ y \in M_0'\big)\,.$$

The operators E, X, and Y are bounded and linear, moreover, $\|Y\| \le 1$. With $\mathscr{L}$ being the differential operator defined in (6) we consider the composition $YS\mathscr{L}XE$:

$$\begin{aligned} C_0^{r+2m}(M_0') \xrightarrow{E} C_0^{r+2m}(W') \xrightarrow{X} C_0^{r+2m}(Q) \\ \xrightarrow{\mathscr{L}} C^r(Q) \xrightarrow{S} L_\infty(M) \xrightarrow{Y} L_\infty(M_0'). \end{aligned}$$

Let

$$\begin{aligned} Z_1 &: C_0^{r+2m}(Q) \to L_\infty(M) \\ Z_2 &: C_0^{r+2m}(W') \to L_\infty(M_0') \end{aligned}$$

be the operators of restriction to M and M_0', respectively. Let, furthermore, $J : C_0^{r+2m}(M_0') \to L_\infty(M_0')$ be the identical embedding. By the definitions,

$$\begin{aligned} S\mathscr{L} &= Z_1 \\ YZ_1X &= Z_2 \\ Z_2E &= J. \end{aligned}$$

This gives

$$YS\mathscr{L}XE = YZ_1XE = Z_2E = J. \tag{74}$$

Next we want to reduce J to S, thus we consider

$$\Lambda = \{\delta_x^\alpha \,:\, x \in (M_0')^0,\ \alpha \in \mathbb{N}_0^{d_1},\ |\alpha| \le r + 2m\}$$

and

$$\widetilde{\Lambda} = \{\delta_x^\alpha \,:\, x \in Q,\ \alpha \in \mathbb{N}_0^d,\ |\alpha| \le r\}.$$

The operator $\mathscr{L}XE$ satisfies

$$\mathscr{L}XE\big(B_{C_0^{r+2m}(M_0')}\big) \subseteq cB_{C^r(Q)} \tag{75}$$

and is of the form (14). To see the latter, consider the mapping $\Phi^{-1} : U \to W'$ and denote its components as follows:

$$\Phi^{-1}(x) = (\theta_1(x), \theta_2(x)),$$

where $\theta_1 : U \to [-1,1]^{d_1}$ and $\theta_2 : U \to [-1,1]^{d-d_1}$ are the respective induced C^∞ mappings (if $d_1 = d$, the second component is dropped). Now we note that for $f \in C_0^{r+2m}(M_0')$

$$(\mathscr{L}XEf)(x) = \begin{cases} \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha\big(f(\theta_1(x))\psi(\theta_2(x))\big) & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \notin U \end{cases}$$

(with $\psi(\theta_2(x))$ replaced by 1 in the case $d_1 = d$). Therefore we have, with $\beta \in \mathbb{N}_0^d$, $|\beta| \le r$, and $x \in Q$

$$\delta_x^\beta(\mathscr{L}XEf) = \begin{cases} D^\beta \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha\big(f(\theta_1(x))\psi(\theta_2(x))\big) & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \notin U. \end{cases}$$

It is readily checked by induction and elementary calculus that there are C^∞ functions h_γ on U ($\gamma \in \mathbb{N}_0^{d_1}$, $|\gamma| \le r + 2m$) such that for all $f \in C_0^{r+2m}(M_0')$

$$\delta_x^\beta(\mathscr{L}XEf) = \begin{cases} \sum_{|\gamma|\le r+2m} h_\gamma(x)(D^\gamma f)(\theta_1(x)) & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \notin U. \end{cases}$$

It follows that $\mathscr{L}XE$ is of the form (14). From Proposition 1 and (75) we obtain

$$e_{c_1 n}^{\mathrm{ran}}\big(YS\mathscr{L}XE, B_{C_0^{r+2m}(M_0')}\big) \le \|Y\| e_n^{\mathrm{ran}}\big(S, cB_{C^r(Q)}\big) \le c e_n^{\mathrm{ran}}\big(S, B_{C^r(Q)}\big),$$

where $c_1 \in \mathbb{N}$ is the number of multiindices $\gamma \in \mathbb{N}_0^{d_1}$ with $|\gamma| \le r + 2m$. Together with (74) this yields,

$$e_{c_1 n}^{\mathrm{ran}}\big(J, B_{C_0^{r+2m}(M_0')}\big) \le c e_n^{\mathrm{ran}}\big(S, B_{C^r(Q)}\big).$$

On the other hand, it is well-known that

$$e_{c_1 n}^{\mathrm{ran}}\big(J, B_{C_0^{r+2m}(M_0')}\big) \ge c n^{-(r+2m)/d_1}.$$

Consequently,

$$e_n^{\mathrm{ran}}\big(S, B_{C^r(Q)}\big) \ge c n^{-(r+2m)/d_1},$$

concluding the proof of the lower bounds.

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