Ultrastability of n-th minimal errors

Stefan Heinrich
Department of Computer Science
University of Kaiserslautern
D-67653 Kaiserslautern, Germany
e-mail: heinrich@informatik.uni-kl.de

Abstract

We use the ultraproduct technique to study local properties of basic quantities of information-based complexity theory – the n-th minimal errors. We consider linear and nonlinear operators in normed spaces, information consists of continuous linear functionals and is assumed to be adaptive. We establish ultrastability and disprove regularity of n-th minimal errors. As a consequence, we answer a question posed by Hinrichs, Novak, and Woźniakowski in a recent paper (*Discontinuous information in the worst case and randomized settings, Math. Nachr., doi:10.1002/mana.201100128*).

1 Introduction

In this paper we apply some techniques from local theory of Banach spaces, in particular ultraproducts, to information-based complexity theory. Our main goal is to understand local properties of basic quantities of this theory – the n-th minimal errors. We consider the deterministic setting with adaptive information consisting of linear functionals.

The central result of this paper is a stability property of the n-th minimal errors with respect to ultraproducts. We present the analysis for arbitrary, in general nonlinear, continuous operators defined on open sets. As an intermediate step towards this we introduce a suitable generalization of the ultraproduct of linear operators to this nonlinear situation. We also provide a counterexample showing that the considered n-th minimal errors are not regular.

Hinrichs, Novak, and Woźniakowski asked in [5], whether the n-th minimal error of a continuous operator is the supremum of the n-th minimal errors of all its restrictions to finite dimensional subspaces. As a consequence of our main result on ultrastability, we obtain the negative answer to this question.

On the other hand, using again ultrastability, we show that the answer is positive if the operator is compact or the target space is 1-complemented in its bidual.

Finally we also discuss the linear case, in which the n-th minimal errors are s-numbers and the results proved can be formulated in terms of s-number properties. Connections of information-based complexity to s-number theory were first explored by Mathé [7].

The paper is organized as follows. In Section 2 we introduce notation and present some basic facts from information-based complexity theory and Banach space ultraproducts. In particular, a suitable notion of the ultraproduct of nonlinear operators is given. Section 3 contains the main result on ultrastability. In section 4 we apply this to various questions of locality of n-th minimal errors and present a counterexample. The final section 5 contains various additional results, in particular the case of linear operators in Banach spaces, as well as a further discussion of ultraproducts of nonlinear operators.

2 Notation

For a normed space (by which we always mean normed linear space) X we let X^* be the dual space, that is, the space of continuous linear functionals on X. Let $\mathcal{B}_X$ be the unit ball of X and, with Y being a normed space, as well, we let $L(X,Y)$ be the space of bounded linear operators from X to Y. For a set $B \subset X$ we denote the interior by B° and the (not necessarily closed) linear hull by $\operatorname{span} B$. The canonical embedding of X into its bidual X^{**} is denoted by K_X. We say that X is 1-complemented in its bidual, if there is a projection of norm one from X^{**} onto $K_X(X) \subset X^{**}$. Finally, we let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We start with notation related to information-based complexity theory. For background we refer to [10, 8]. Information-based complexity theory is aiming at investigating general classes of algorithms for computational problems of analysis, finding algorithms of optimal behaviour, minimal possible errors, lower bounds, and understanding the complexity, that is, the intrinsic difficulty of such computational problems.

Let us first give an informal description. The goal of an algorithm is to approximate the solution $S(x) \in Y$ of a numerical problem, represented by a mapping $S : F \to Y$, where $F \subset X$ is a subset, at input $x \in F$. The algorithm can access x only by evaluating a limited number of linear functionals.

One of the basic approaches to a general notion of an algorithm is the following. The algorithm starts with evaluating a functional $L_1 \in X^*$ at the input x, that is $L_1(x)$. Depending on this value, another functional $L_2 \in X^*$ is chosen and $L_2(x)$ is evaluated. The choice of the next functional $L_3 \in X^*$ may depend on $L_1(x)$ and $L_2(x)$, and so on. The procedure goes on until n values $L_j(x)$ ($j = 1, \dots, n$) are obtained, the 'information' about x. On the basis of this information a final mapping $\varphi : \mathbb{R}^n \to Y$ is applied, representing the computations on the information leading to the approximation to $S(x)$ in Y. This is formalized as follows.

For a normed space X and $n \in \mathbb{N}$ we first define $\mathcal{N}_n^{\mathrm{ad}}(X)$. An element $N \in \mathcal{N}_n^{\mathrm{ad}}(X)$ is a tuple $N = (L_1, \ldots, L_n)$, where

$$L_1 \in X^*$$

and for $2 \leq k \leq n$,

$$L_k : X \times \mathbb{R}^{k-1} \to \mathbb{R}$$

is a function such that for all $(a_1, \ldots, a_{k-1}) \in \mathbb{R}^{k-1}$

$$L_k(\,\cdot\,, a_1, \ldots, a_{k-1}) \in X^*.$$

Given $N \in \mathcal{N}_n^{\mathrm{ad}}(X)$, we associate with it a mapping $N : X \to \mathbb{R}^n$ (we use the same letter) as follows. For $x \in X$ put

$$\begin{aligned} L_1(x) &= a_1 \\ L_i(x, a_1, \ldots, a_{i-1}) &= a_i \quad (2 \leq i \leq n) \end{aligned}$$

and

$$N(x) = (a_1, a_2, \ldots, a_n).$$

Let $\Phi_n(Y)$ be the set of all mappings $\varphi : \mathbb{R}^n \to Y$. Given any nonempty set $F \subset X$, another normed space Y, and an arbitrary mapping $S : F \to Y$, we define for $N \in \mathcal{N}_n^{\mathrm{ad}}(X)$ and $\varphi \in \Phi_n(Y)$

$$e(S, \varphi \circ N, F, Y) = \sup_{x \in F} \|S(x) - \varphi(N(x))\|,$$

which is the error of $\varphi \circ N$ as an approximation of S on F. For $n \in \mathbb{N}_0$ the n-th minimal error is defined as follows. If $n = 0$, we put

$$e_0(S, F, X, Y) = \inf_{y \in Y} \sup_{x \in F} \|S(x) - y\|,$$

and if $n \geq 1$, we set

$$e_n(S, F, X, Y) = \inf_{N \in \mathcal{N}_n^{\mathrm{ad}}(X), \varphi \in \Phi_n(Y)} e(S, \varphi \circ N, F, Y).$$

These quantities play a crucial role in lower bound proofs of information-based complexity theory. Indeed, it follows from the definition that no algorithm for the approximation of S on F that uses n linear functionals can have a smaller error than $e_n(S, F, X, Y)$. Let us note some simple properties, which we need later on.

If X is a (linear, not necessarily closed) subspace of a normed space $\tilde{X}$, then for each $N \in \mathcal{N}_n^{\mathrm{ad}}(X)$ there exists an $\tilde{N} \in \mathcal{N}_n^{\mathrm{ad}}(\tilde{X})$ with

$$\tilde{N}(x) = N(x) \quad (x \in X). \tag{1}$$

Indeed, if $N = (L_1, \dots, L_n)$, we define $\tilde{N} = (\tilde{L}_1, \dots, \tilde{L}_n)$ in such a way that $\tilde{L}_k(\,\cdot\, a_1, \dots, a_{k-1})$ is any continuous linear extension of $L_k(\,\cdot\, a_1, \dots, a_{k-1})$ to all of $\tilde{X}$ (e.g., by the Hahn-Banach theorem). Therefore we have

$$e(S, \varphi \circ N, F, Y) = e(S, \varphi \circ \tilde{N}, F, Y) \tag{2}$$

for all $\varphi \in \Phi_n(Y)$. Conversely, if we start with any $\tilde{N} \in \mathcal{N}_n^{\mathrm{ad}}(\tilde{X})$ and let $N \in \mathcal{N}_n^{\mathrm{ad}}(X)$ be obtained by restriction of $\tilde{L}(\,\cdot\,, a_1, \dots, a_{k-1})$ to X, then (1) and (2) hold again. It follows that

$$e_n(S, F, X, Y) = e_n(S, F, \tilde{X}, Y), \tag{3}$$

so the n-th minimal error depends only on $\operatorname{span} F$ (endowed with the induced norm), not on the particular superspace containing $\operatorname{span} F$. As a consequence, we drop the indication of the source space X in the notation $e_n(S, F, X, Y)$ and write $e_n(S, F, Y)$ in the sequel.

Concerning the target space, let us denote the completion of Y by $\hat{Y}$. Then it is obvious from the definition that

$$e(S, F, Y) = e(S, F, \hat{Y}).$$

On the basis of these remarks we may assume without loss of generality (and do so in Section 5.3) that X and Y are Banach spaces.

Next suppose $N \in \mathcal{N}_n^{\mathrm{ad}}(X)$ and $U \in L(X_1, X)$ with X_1 another normed space. Then we can define a new information operator $N \circ U = (\tilde{L}_1, \dots, \tilde{L}_n)$ by setting

$$\tilde{L}_1(x_1) = L_1(Ux_1) \quad (x_1 \in X_1). \tag{4}$$

and for $2 \le k \le n$ and $a_1, \dots, a_{k-1} \in \mathbb{R}^{k-1}$

$$\tilde{L}_k(x_1, a_1, \dots, a_{k-1}) = L_k(Ux_1, a_1, \dots, a_{k-1}) \quad (x_1 \in X_1). \tag{5}$$

It is readily checked that $N \circ U \in \mathcal{N}_n^{\mathrm{ad}}(X_1)$ and

$$(N \circ U)(x_1) = N(Ux_1) \quad (x_1 \in X_1).$$

Lemma 2.1. *Let X, Y, S, F be as above, let X_1, Y_1 be normed spaces, $U \in L(X_1, X)$, $\emptyset \ne F_1 \subset X_1$ a subset with $U(F_1) \subset F$, and let $V : Y \to Y_1$ be a mapping with Lipschitz constant $\|V\|_{\mathrm{Lip}} < \infty$. Then for all $N \in \mathcal{N}_n^{\mathrm{ad}}(X)$ and $\varphi \in \Phi_n(Y)$*

$$e(VSU, (V \circ \varphi) \circ (N \circ U), F_1, Y_1) \le \|V\|_{\mathrm{Lip}}\, e(S, \varphi \circ N, F, Y). \tag{6}$$

Consequently,

$$e_n(VSU, F_1, Y_1) \le \|V\|_{\mathrm{Lip}}\, e_n(S, F, Y). \tag{7}$$

If $V \in L(Y, Y_1)$ is an isometry, that is $\|Vx\| = \|x\|$ $(x \in X)$, then

$$e_n(S, F, Y) \le 2 e_n(VS, F, Y_1). \tag{8}$$

Finally, if S is linear and $\lambda \in \mathbb{R}$, then

$$e_n(\lambda S, F, Y) = e_n(S, \lambda F, Y) = |\lambda| e_n(S, F, Y). \tag{9}$$

Proof. We have

$$\begin{aligned} & e(VSU, (V \circ \varphi) \circ (N \circ U), F_1, Y_1) \\ = & \sup_{x_1 \in F_1} \|VS(Ux_1) - V\varphi(N(Ux_1))\| \\ \leq & \|V\|_{\mathrm{Lip}} \sup_{x_1 \in F_1} \|S(Ux_1) - \varphi(N(Ux_1))\| \\ \leq & \|V\|_{\mathrm{Lip}} \sup_{x \in F} \|S(x) - \varphi(N(x))\|, \end{aligned}$$

which proves (6) and hence (7).

To prove (8), let $\varepsilon > 0$ and let $N \in \mathcal{N}_n^{\mathrm{ad}}(X)$, $\varphi_1 \in \Phi_n(Y_1)$ be such that

$$\sup_{x \in F} \|VS(x) - \varphi_1(N(x))\| \leq e_n(VS, F, Y_1) + \varepsilon.$$

For $a \in N(F)$ we take any $x_a \in F$ with $N(x_a) = a$ and put $\varphi(a) = S(x_a)$. For $a \in \mathbb{R}^n \setminus N(F)$ we put $\varphi(a) = 0 \in Y$. This defines $\varphi \in \Phi_n(Y)$. Now let $x \in F$ and $a = N(x)$. Then

$$\begin{aligned} & \|S(x) - \varphi(N(x))\| \\ = & \|S(x) - S(x_a)\| = \|VS(x) - VS(x_a)\| \\ \leq & \|VS(x) - \varphi_1(N(x))\| + \|VS(x_a) - \varphi_1(N(x_a))\| \\ \leq & 2e_n(VS, F, Y_1) + 2\varepsilon, \end{aligned}$$

which proves (8).

If S is linear and $\lambda \in \mathbb{R}$, then we conclude, using (7) repeatedly,

$$\begin{aligned} |\lambda| e_n(S, F, Y) &= |\lambda| e_n(\lambda^{-1}\lambda S, F, Y) \leq e_n(\lambda S, F, Y) \\ &\leq e_n(S, \lambda F, Y) = e_n(\lambda^{-1}\lambda S, \lambda F, Y) \\ &\leq e_n(\lambda S, F, Y)| \leq |\lambda| e_n(S, F, Y). \end{aligned}$$

□

If $F \subset X$ is absolutely convex and $S : X \to Y$ is linear, we define for $n \in \mathbb{N}$

$$\begin{aligned} c_0(S, F, Y) &= \sup_{x \in F} \|S(x)\|, \\ c_n(S, F, Y) &= \inf_{f_1, \ldots, f_n \in X^*} \sup_{x \in F, f_1(x) = \ldots = f_n(x) = 0} \|S(x)\| \quad (n \geq 1). \end{aligned}$$

If X and Y are Banach spaces, $F = \mathcal{B}_X$ and $S \in L(X, Y)$, then $c_n(S, \mathcal{B}_X, Y)$ is the n-th Gelfand number of S. For the following result we refer to [10], ch. 5.4.

Lemma 2.2. *Let $F \subset X$ be absolutely convex and $S : X \to Y$ linear. Then*

$$c_n(S, F, Y) \leq e_n(S, F, Y) \leq 2c_n(S, F, Y).$$

Moreover, if $Y = \ell_\infty(D)$ for some set D, then

$$e_n(S, F, Y) = c_n(S, F, Y). \tag{10}$$

Hence, if $F \subset X$ is absolutely convex and $S : X \to Y$ is linear, we have

$$e_n(S, F, Y) \leq 2c_n(S, F, Y) = 2c_n(K_Y S, F, Y^{**}) \leq 2e_n(K_Y S, F, Y^{**}).$$

Now let us turn to ultraproducts. For background on filters and ultrafilters we refer to [2], for Banach space ultraproducts to [4]. Ultrafilters and ultraproducts are an elegant and convenient way of handling various compactness arguments.

Let us briefly review some notions. A filter $\mathcal{F}$ on a nonempty set I is a set of nonempty subsets of I such that $I_1, I_2 \in \mathcal{F}$ implies $I_1 \cap I_2 \in \mathcal{F}$ and $I_1 \in \mathcal{F}$ implies $I_2 \in \mathcal{F}$ for any superset $I_2 \supseteq I_1$. A filter $\mathcal{F}_2$ dominates a filter $\mathcal{F}_1$ if $\mathcal{F}_1 \subset \mathcal{F}_2$. An ultrafilter is a filter $\mathcal{U}$ such that each filter dominating $\mathcal{U}$ coincides with $\mathcal{U}$. Each filter is dominated by some ultrafilter. Let us note that this statement, which is basic to our paper, requires the axiom of choice (via Zorn's Lemma). Given $I_0 \in \mathcal{U}$, we let

$$\mathcal{U}|I_0 = \{I_1 \in \mathcal{U} : I_1 \subset I_0\}$$

be the induced ultrafilter on I_0. An ultrafilter $\mathcal{U}$ is called countably incomplete, if there is a sequence $(I_n)_{n=1}^\infty$ with $I_n \in \mathcal{U}$ and $\cap_{n=1}^\infty I_n = \emptyset$.

Ultrafilters have the following properties. Given an arbitrary set $I_0 \subset I$, then either $I_0 \in \mathcal{U}$ or $I \setminus I_0 \in \mathcal{U}$. For $t_i, t \in T$ $(i \in I)$ with T a topological space, we write

$$t = \lim_{\mathcal{U}} t_i$$

if $\{i \in I : t_i \in V\} \in \mathcal{U}$ for each neighborhood V of t. If T is compact, then for each family $(t_i)_{i \in I} \subset T$ there exists a $t \in T$ such that $t = \lim_{\mathcal{U}} t_i$. This is the key property for various compactness arguments ultrafilters and ultraproducts are used in.

Given a family of normed spaces $(X_i)_{i \in I}$, we denote by $\ell_\infty(I, X_i)$ the normed space of all families $(x_i)_{i \in I}$ with $x_i \in X_i$ and

$$\|(x_i)_{i \in I}\|_{\ell_\infty(I, X_i)} = \sup_{i \in I} \|x_i\| < \infty.$$

For an ultrafilter $\mathcal{U}$ on I, we define the ultraproduct $(X_i)_{\mathcal{U}}$ as the set of all equivalence classes $(x_i)_{\mathcal{U}}$ of families $(x_i)_{i \in I} \in \ell_\infty(I, X_i)$ under the equivalence relation

$$(x_i)_{i \in I} \sim_{\mathcal{U}} (y_i)_{i \in I} \quad \text{iff} \quad \lim_{\mathcal{U}} \|x_i - y_i\| = 0.$$

Equipped with the norm

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|,$$

$(X_i)_{\mathcal{U}}$ becomes a normed space. If all X_i are Banach spaces, then $(X_i)_{\mathcal{U}}$ is a Banach space. The ultraproduct of the dual spaces $(X_i^*)_{\mathcal{U}}$ can be identified with a subspace of $(X_i)_{\mathcal{U}}^*$ by setting for $f = (f_i)_{\mathcal{U}} \in (X_i^*)_{\mathcal{U}}$ and $x = (x_i)_{\mathcal{U}} \in (X_i)_{\mathcal{U}}$

$$f(x) = \lim_{\mathcal{U}} f_i(x_i).$$

If $X_i \equiv X$, the ultraproduct is called an ultrapower and is denoted by $(X)_{\mathcal{U}}$.

Let X_i, Y_i be normed spaces and $S_i \in L(X_i, Y_i)$ $(i \in I)$ be bounded linear operators satisfying

$$\sup_I \|S_i\| < \infty.$$

Then the ultraproduct $(S_i)_{\mathcal{U}} \in L((X_i)_{\mathcal{U}}, (Y_i)_{\mathcal{U}})$ is defined for $(x_i)_{\mathcal{U}} \in (X_i)_{\mathcal{U}}$ by

$$(S_i)_{\mathcal{U}}(x_i)_{\mathcal{U}} = (S_i x_i)_{\mathcal{U}}.$$

If $X_i \equiv X$, $Y_i \equiv Y$, and $S_i \equiv S$, we write $(S)_{\mathcal{U}} \in L((X)_{\mathcal{U}}, (Y)_{\mathcal{U}})$.

Now we generalize this to nonlinear mappings defined on subsets (compare [1], chapter 2.V). Let $\emptyset \neq F_i \subset X_i$ and let $S_i : F_i \to Y_i$ be arbitrary mappings $(i \in I)$. We let

$$\mathcal{D}((S_i, F_i), \mathcal{U}) \subset (X_i)_{\mathcal{U}}$$

– the domain of definition of the ultraproduct – be the set of all $x \in (X_i)_{\mathcal{U}}$ such that there exists a family $(x_i) \in \ell_\infty(I, X_i)$ with $(x_i)_{\mathcal{U}} = x$,

$$\{i \in I : \, x_i \in F_i\} \quad \in \quad \mathcal{U}, \tag{11}$$

$$\lim_{\mathcal{U}|\{i \in I : \, x_i \in F_i\}} \|S_i(x_i)\| \quad < \quad \infty, \tag{12}$$

and for each family $(z_i) \in \ell_\infty(I, X_i)$ with $(z_i)_{\mathcal{U}} = x$ and $\{i \in I : \, z_i \in F_i\} \in \mathcal{U}$ we have

$$\lim_{\mathcal{U}|\{i \in I : \, x_i, z_i \in F_i\}} \|S_i(x_i) - S_i(z_i)\| = 0. \tag{13}$$

We note that

$$\mathcal{D}((S_i, F_i), \mathcal{U}) \subset (\operatorname{span} F_i)_{\mathcal{U}}. \tag{14}$$

Clearly, $\mathcal{D}((S_i, F_i), \mathcal{U})$ could be empty. If this is not the case, we define the ultraproduct

$$(S_i)_{\mathcal{U}} : \mathcal{D}((S_i, F_i), \mathcal{U}) \to (Y_i)_{\mathcal{U}}$$

as follows. For $x \in \mathcal{D}((S_i, F_i), \mathcal{U})$ with $x = (x_i)_{\mathcal{U}}$ being any representation satisfying (11) we put

$$(S_i)_{\mathcal{U}}(x) = (y_i)_{\mathcal{U}}, \quad y_i = \begin{cases} S_i(x_i) & \text{if} \quad x_i \in F_i \\ 0 & \text{otherwise.} \end{cases}$$

The definition of $\mathcal{D}((S_i, F_i), \mathcal{U})$ ensures that $(S_i)_{\mathcal{U}}$ is well-defined.

For our applications to information-based complexity we slightly restrict the domain of definition (we comment on the relation of both domains in Section 5.2). We let

$$\mathcal{D}_0((S_i, F_i), \mathcal{U}) \subset \mathcal{D}((S_i, F_i), \mathcal{U})$$

be the set of all $x \in \mathcal{D}((S_i, F_i), \mathcal{U})$ with the following additional property: For each family $(x_i) \in \ell_\infty(I, \operatorname{span} F_i)$ with $(x_i)_{\mathcal{U}} = x$ we have

$$\{i \in I : \, x_i \in F_i\} \in \mathcal{U}. \tag{15}$$

The above definition for uniformly bounded linear operators is a special case with $F_i = X_i$ and $\mathcal{D}_0((S_i, F_i),\mathcal{U}) = \mathcal{D}((S_i, F_i),\mathcal{U}) = (X_i)_{\mathcal{U}}$. Note that if $F_i = \mathcal{B}^\circ_{X_i}$, then

$$\mathcal{D}_0((S_i, F_i),\mathcal{U}) \subset \mathcal{B}^\circ_{(X_i)_{\mathcal{U}}}.$$

Finally, if $X_i \equiv X$, $Y_i \equiv Y$, $F_i \equiv F$, $S_i \equiv S : F \to Y$, we write $(S)_{\mathcal{U}}$ and $\mathcal{D}((S, F),\mathcal{U})$, resp. $\mathcal{D}_0((S, F),\mathcal{U})$. If F is open and S is continuous, then

$$J(F) \subset \mathcal{D}_0((S, F),\mathcal{U}), \tag{16}$$

where J is the canonical embedding of X into $(X)_{\mathcal{U}}$ given by

$$Jx = (x)_{\mathcal{U}} \quad (x \in X). \tag{17}$$

We refer to Section 5.2 for further details on the ultraproduct of nonlinear operators.

Let us recall the principle of local reflexivity [6, 3], which we will apply several times.

Lemma 2.3. *Let X be a normed space, $E \subset X^{**}$ a finite dimensional subspace, $n \in \mathbb{N}$, $f_1, \dots, f_n \in X^*$, and $\varepsilon > 0$. Then there is an invertible linear operator T from E onto a subspace of X such that $\|T\|\|T^{-1}\| \le 1 + \varepsilon$, $Tx = x$ for all $x \in E \cap X$ and $(Tu, f_k) = (u, f_k)$ for all $u \in E$, $k = 1, \dots, n$.*

This principle is usually stated for X being a Banach space, but the case of a normed space X follows readily from the statement for the completion $\hat{X}$ of X by noting that for $f_1, \dots, f_n \in X^*$ $(= \hat{X}^*)$ and $a_1, \dots, a_n \in \mathbb{R}$ the set

$$\{x \in X : \ f_1(x) = a_1, \dots, f_n(x) = a_n\}$$

is dense in

$$\{x \in \hat{X} : \ f_1(x) = a_1, \dots, f_n(x) = a_n\}.$$

3 Ultrastability

In this section we prove the central result of this paper. The following two lemmas, which are of geometric nature, serve as preparations. The first lemma shows that an arbitrary information operator can be replaced equivalently by an information operator possessing certain uniformity properties (required later on for taking ultraprocucts).

Lemma 3.1. *Let $n \in \mathbb{N}$, let X be a normed space with $\dim X \ge n$, let $0 < \delta < 1$ and $M \in \mathcal{N}^{\mathrm{ad}}_n(X)$. Then there exists $N = (L_1, \dots, L_n) \in \mathcal{N}^{\mathrm{ad}}_n(X)$ such that the following hold: $\|L_1\| = 1$ and for all $a_1, \dots, a_{n-1} \in \mathbb{R}$ and $1 < k \le n$*

$$\begin{aligned} \|L_k(\,\cdot\,, a_1, \dots, a_{k-1})\| &= 1 & (18)\\ \operatorname{dist}(L_k(\,\cdot\,, a_1, \dots, a_{k-1}), E_{k-1}) &= 1, & (19) \end{aligned}$$

where

$$E_{k-1} = \operatorname{span}(L_1, L_2(\,\cdot\,, a_1), \dots, L_{k-1}(\,\cdot\,, a_1, \dots, a_{k-2})).$$

Moreover, there is a mapping $\psi : \mathbb{R}^n \to \mathbb{R}^n$ *such that for all* $x \in X$

$$M(x) = \psi(N(x)). \tag{20}$$

Proof. We argue by induction over n. Let $n = 1$, $M = M_1$. If $M_1 \neq 0$ we put $L_1 = \|M_1\|^{-1} M_1$ and if $M_1 = 0$, we let $L_1 \in X^*$ be any element of norm 1. In both cases we set $\psi_1(a_1) = \|M_1\| a_1$. Obviously, (20) is satisfied.

Now let $n > 1$ and assume that the statement is correct for $n - 1$. Let $M \in \mathcal{N}_n^{\mathrm{ad}}(X)$, $M = (M_1, \dots, M_n)$. Clearly, $\tilde{M} = (M_1, \dots, M_{n-1}) \in \mathcal{N}_{n-1}^{\mathrm{ad}}(X)$. By assumption, there is an $\tilde{N} = (L_1, \dots, L_{n-1}) \in \mathcal{N}_{n-1}^{\mathrm{ad}}(X)$ and a $\tilde{\psi} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that the statement of the lemma holds for $\tilde{M}, \tilde{N}, \tilde{\psi}$.

Let $a_1, \dots, a_n \in \mathbb{R}$ and let

$$\begin{aligned} g_1 &= L_1 \\ g_j &= L_j(\,\cdot\,, a_1, \dots, a_{j-1}) \quad (2 \leq j \leq n-1). \end{aligned}$$

Put

$$(b_1, \dots, b_{n-1}) = \tilde{\psi}(a_1, \dots, a_{n-1})$$

and define

$$f_n = M_n(\,\cdot\,, b_1, \dots, b_{n-1}).$$

We consider two cases. If

$$f_n \notin \operatorname{span}(g_1, \dots, g_{n-1}),$$

we choose $g_n \in \operatorname{span}(g_1, \dots, g_{n-1}, f_n)$ with

$$\|g_n\| = 1 \quad \text{and} \quad \operatorname{dist}(g_n, \operatorname{span}(g_1, \dots, g_{n-1})) = 1. \tag{21}$$

On the other hand, if

$$f_n \in \operatorname{span}(g_1, \dots, g_{n-1}),$$

we let g_n be any element of X^* satisfying (21). In both cases there are $d_1, \dots, d_n \in \mathbb{R}$ such that

$$f_n = \sum_{j=1}^{n} d_j g_j.$$

Now we define

$$\begin{aligned} L_n(\,\cdot\,, a_1, \dots, a_{n-1}) &= g_n \\ N &= (L_1, \dots, L_{n-1}, L_n) \\ \psi(a_1, \dots, a_n) &= \Big(\tilde{\psi}(a_1, \dots, a_{n-1}), \sum_{j=1}^{n} d_j a_j\Big). \end{aligned}$$

Properties (18) and (19) follow from the construction. It remains to show (20). Let $x \in X$ and put

$$\begin{aligned} a_1 &= L_1(x) \\ a_j &= L_j(x, , a_1, \dots, a_{j-1}) \quad (2 \le j \le n) \end{aligned}$$

$$\begin{aligned} b_1 &= M_1(x) \\ b_j &= M_j(x, , b_1, \dots, b_{j-1}) \quad (2 \le j \le n). \end{aligned}$$

By the induction assumption,

$$(b_1, \dots, b_{n-1}) = \tilde{\psi}(a_1, \dots, a_{n-1}),$$

while by construction

$$b_n = M_n(x, b_1, \dots, b_{n-1}) = f_n(x) = \sum_{j=1}^{n} d_j g_j(x) = \sum_{j=1}^{n} d_j a_j,$$

so

$$(b_1, \dots, b_{n-1}, b_n) = \Big(\tilde{\psi}(a_1, \dots, a_{n-1}), \sum_{j=1}^{n} d_j a_j\Big) = \psi(a_1, \dots, a_n).$$

□

The next lemma is a simple geometric fact on the existence of biorthogonal sequences with uniform norm bounds.

Lemma 3.2. *Let X be a normed space, $0 < \delta \le 1$, $n \in \mathbb{N}$, and let $f_1, \dots, f_n \in X^*$ be such that $\|f_k\| = 1$ $(1 \le k \le n)$ and for $1 < k \le n$*

$$\operatorname{dist}(f_k, \operatorname{span}(f_1, \dots, f_{k-1})) \ge \delta.$$

Then for each $\varepsilon > 0$ there exist $x_1, \dots, x_n \in X$ such that

$$f_j(x_k) = \delta_{jk} \quad \textit{and} \quad \|x_k\| \le (1+\varepsilon)(1+\delta^{-1}+\varepsilon)^{n-k} \quad (1 \le j, k \le n).$$

Proof. We use induction over n. The case $n = 1$ is obvious. Assuming that $n \ge 2$ and the statement holds for $n-1$, we find $z_1, \dots, z_{n-1}$ such that $f_j(z_k) = \delta_{jk}$ $(1 \le j, k \le n-1)$ and

$$\|z_k\| \le (1+\varepsilon)(1+\delta^{-1}+\varepsilon)^{n-1-k}.$$

Consider the functional h on $\operatorname{span}(f_1, \dots, f_n)$ defined by

$$h(f_k) = 0 \quad (1 \le k \le n-1), \quad h(f_n) = 1.$$

Then $\|h\| \le \delta^{-1}$. Extend h to all of X^* with preservation of the norm and use the local reflexivity Lemma 2.3 to find an $x_n \in X$ such that

$$\|x_n\| \le \delta^{-1} + \varepsilon$$

and

$$f_k(x_n) = 0 \quad (1 \le k \le n-1), \quad f_n(x_n) = 1.$$

Now we put for $1 \le k \le n-1$

$$x_k = z_k - f_n(z_k)x_n.$$

Hence, for $1 \le j, k \le n$ we have $f_j(x_k) = \delta_{jk}$ and

$$\|x_k\| \le \|z_k\|(1 + \|x_n\|) \le (1+\varepsilon)(1+\delta^{-1}+\varepsilon)^{n-k}.$$

□

Now we are ready to state the main result of this paper, which shows that the n-th minimal errors are ultrastable, meaning that the n-th minimal error of an ultraproduct is bounded from above by the limit of the n-th minimal errors of the factors. This result will have numerous applications, most of them to be discussed in the next section.

Theorem 3.3. *Let I be a nonempty set, let X_i, Y_i be normed spaces, $\emptyset \ne F_i \subset X_i$ arbitrary subsets, $S_i : F_i \to Y_i$ arbitrary mappings $(i \in I)$, and let $\mathcal{U}$ be an ultrafilter on I. Assume that $\mathcal{D}_0((S_i, F_i), \mathcal{U}) \ne \emptyset$. Then for all $n \in \mathbb{N}_0$*

$$e_n\big((S_i)_{\mathcal{U}}, \mathcal{D}_0((S_i, F_i), \mathcal{U}), (Y_i)_{\mathcal{U}}\big) \le \lim_{\mathcal{U}} e_n(S_i, F_i, Y_i). \tag{22}$$

If, moreover, $\mathcal{U}$ is countably incomplete, then for each n there exist $N \in \mathcal{N}_n^{\mathrm{ad}}((X_i)_{\mathcal{U}})$ and $\varphi \in \Phi_n((Y_i)_{\mathcal{U}})$ such that

$$e\big((S_i)_{\mathcal{U}}, \varphi \circ N, \mathcal{D}_0((S_i, F_i), \mathcal{U}), (Y_i)_{\mathcal{U}}\big) \le \lim_{\mathcal{U}} e_n(S_i, F_i, Y_i). \tag{23}$$

Proof. If $\lim_{\mathcal{U}} e_n(S_i, F_i, Y_i) = \infty$, then the result holds trivially. So we suppose

$$\lim_{\mathcal{U}} e_n(S_i, F_i, Y_i) < \infty. \tag{24}$$

Furthermore, we can assume

$$\operatorname{span} F_i = X_i \quad (i \in I), \tag{25}$$

since enlarging the source space affects none of the quantities involved in (22) or (23), see (1–3) and (14).

If $\lim_{\mathcal{U}} \dim X_i < n$, then $\{i \in I : \dim X_i < n\} \in \mathcal{U}$ and $\dim (X_i)_{\mathcal{U}} < n$. It readily follows that both sides of (22) are zero. Thus, we suppose

$$\lim_{\mathcal{U}} \dim X_i \ge n.$$

Then we can assume without loss of generality that $\dim X_i \geq n$ for all $i \in I$, since changing the factors on a set $I_1 \notin \mathcal{U}$ does not affect the ultraproduct (of spaces and operators). For each $i \in I$, let $0 < \varepsilon_i \leq 1$ (to be specified later), $N_i = (L_{1,i}, \ldots, L_{n,i}) \in \mathcal{N}_n^{\mathrm{ad}}(X_i)$ and $\varphi_i \in \Phi_n(Y_i)$ with

$$e(S_i, \varphi_i \circ N_i, F_i, Y_i) \leq e_n(S_i, F_i, Y_i) + \varepsilon_i, \tag{26}$$

where we assume the N_i to satisfy the properties in Lemma 3.1. Define $N = (L_1, \ldots, L_n) \in \mathcal{N}_n^{\mathrm{ad}}\big((X_i)_{\mathcal{U}}\big)$ as follows. For $k = 1$ we set

$$L_1 = (L_{1,i})_{\mathcal{U}} \in (X_i)_{\mathcal{U}}^* \tag{27}$$

and for $k > 1$ and $a_1, \ldots, a_{k-1} \in \mathbb{R}$

$$L_k = \big(L_{k,i}(\,\cdot\,, a_1, \ldots, a_{k-1})\big)_{\mathcal{U}} \in (X_i)_{\mathcal{U}}^*. \tag{28}$$

Next we define $\varphi \in \Phi_n((Y_i)_{\mathcal{U}})$. Let $a \in \mathbb{R}^n$. If $\lim_{\mathcal{U}} \|\varphi_i(a)\| < \infty$, we set

$$I_a = \{i \in I : \ \|\varphi_i(a)\| \leq \lim_{\mathcal{U}} \|\varphi_i(a)\| + 1\}$$

and

$$\varphi(a) = (y_i)_{\mathcal{U}}, \quad y_i = \begin{cases} \varphi_i(a) & \text{if} \quad i \in I_a, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

If $\lim_{\mathcal{U}} \|\varphi_i(a)\| = \infty$, we put

$$\varphi(a) = 0.$$

Now let $x = (x_i)_{\mathcal{U}} \in \mathcal{D}_0((S_i, F_i), \mathcal{U})$ and $N(x) = a = (a_1, \ldots, a_n)$. By (27) and (28)

$$\lim_{\mathcal{U}} L_{1,i}(x_i) \;=\; a_1 \tag{30}$$

$$\lim_{\mathcal{U}} L_{k,i}(x_i, a_1, \ldots, a_{k-1}) \;=\; a_k \quad (2 \leq k \leq n). \tag{31}$$

Define for $1 \leq k \leq n$, $i \in I$

$$\beta_{1,i} \;=\; a_1 - L_{1,i}(x_i) \tag{32}$$

$$\beta_{k,i} \;=\; a_k - L_{k,i}(x_i, a_1, \ldots, a_{k-1}) \quad (2 \leq k \leq n). \tag{33}$$

By the assumptions on N_i,

$$\sup_{i \in I} |\beta_{k,i}| \leq |a_k| + \sup_{i \in I} \|x_i\| < \infty. \tag{34}$$

Put

$$f_{1,i} \;=\; L_{1,i} \in X_i^* \tag{35}$$

$$f_{k,i} \;=\; L_{k,i}(\,\cdot\, a_1, \ldots, a_{k-1}) \in X_i^* \quad (2 \leq k \leq n). \tag{36}$$

Again by our assumptions on the N_i we can apply Lemma 3.2 with $\varepsilon = \delta = 1$ to find $z_{k,i} \in X_i$ such that

$$f_{j,i}(z_{k,i}) = \delta_{jk} \quad \text{and} \quad \|z_{k,i}\| \le 2 \cdot 3^{n-k} \quad (1 \le j, k \le n).$$

Define

$$v_i = \sum_{k=1}^{n} \beta_{k,i} z_{k,i}.$$

Then

$$f_{k,i}(v_i) = \beta_{k,i} \tag{37}$$

and

$$\|v_i\| \le 2 \sum_{k=1}^{n} 3^{n-k} |\beta_{k,i}|. \tag{38}$$

It follows from (32–37) that

$$L_{1,i}(x_i + v_i) = a_1 \tag{39}$$

$$L_{k,i}(x_i + v_i, a_1, \dots, a_{k-1}) = a_k \quad (2 \le k \le n). \tag{40}$$

Moreover, (34) and (38) imply that $\sup_{i \in I} \|v_i\| < \infty$, and from (30–33) and (38) we conclude $\lim_{\mathcal{U}} \|v_i\| = 0$. Consequently,

$$(x_i + v_i)_{\mathcal{U}} = (x_i)_{\mathcal{U}} = x \in \mathcal{D}_0((S_i, F_i), \mathcal{U}). \tag{41}$$

By (15), (25), and (41),

$$I_0 := \{i \in I : x_i + v_i \in F_i\} \in \mathcal{U}.$$

Moreover, by (39–40), $N_i(x_i + v_i) = a$. Thus (26) gives for $i \in I_0$

$$\|S_i(x_i + v_i) - \varphi_i(a)\| \le e_n(S_i, F_i, Y_i) + \varepsilon_i. \tag{42}$$

Therefore we get from (12), (13), and (41)

$$\lim_{\mathcal{U}|I_0} \|S_i(x_i + v_i)\| < \infty.$$

This together with (24) and (42) implies $\lim_{\mathcal{U}} \|\varphi_i(a)\| < \infty$, and we conclude from (29), (41), and (42) that

$$\begin{aligned} \|(S_i)_{\mathcal{U}}(x) - \varphi(a)\| &= \lim_{\mathcal{U}|I_0} \|S_i(x_i + v_i) - \varphi_i(a)\| \\ &\le \lim_{\mathcal{U}} e_n(S_i, F_i, Y_i) + \lim_{\mathcal{U}} \varepsilon_i. \end{aligned}$$

Hence,

$$e\big((S_i)_{\mathcal{U}}, \varphi \circ N, \mathcal{D}_0((S_i, F_i), \mathcal{U}), (Y_i)_{\mathcal{U}}\big) \le \lim_{\mathcal{U}} e_n(S_i, F_i, Y_i) + \lim_{\mathcal{U}} \varepsilon_i.$$

If $\mathcal{U}$ is arbitrary, we take any $\varepsilon > 0$ and put $\varepsilon_i \equiv \varepsilon$, which yields (22). If $\mathcal{U}$ is countably incomplete, then we let $(I_k)_{k=1}^{\infty}$ be such that $I_1 \supset I_2 \supset \dots$, $I_k \in \mathcal{U}$ and $\cap_{k=1}^{\infty} I_k = 0$. Now we set $\varepsilon_i = 1$ for $i \notin I_1$ and $\varepsilon_i = 1/k$ for $i \in I_k \setminus I_{k+1}$ $(k = 1, 2, \dots)$. This gives $\lim_{\mathcal{U}} \varepsilon_i = 0$ and (23) follows. □

Let us mention a first consequence of Theorem 3.3, which shows that under quite general assumptions the n-th minimal error is attained.

Corollary 3.4. *Let X, Y be normed spaces, $\emptyset \neq F \subset X$ open, $S : F \to Y$ continuous, and assume that Y is 1-complemented in Y^{**}. Then for each $n \in \mathbb{N}$ there exist $N \in \mathcal{N}_n^{\mathrm{ad}}(X)$ and $\varphi \in \Phi_n(Y)$ such that $\varphi \circ N$ attains the n-th minimal error, i.e.,*

$$e(S, \varphi \circ N, F, Y) = e_n(S, F, Y). \tag{43}$$

Proof. Let $P : Y^{**} \to Y$ be a projection with $\|P\| = 1$. Let $\mathcal{U}$ be any non-trivial ultrafilter on $\mathbb{N}$ (meaning that $\mathcal{U}$ is not generated by a one-element set), hence $\mathcal{U}$ is countably incomplete. Let $J : X \to (X)_{\mathcal{U}}$ be the embedding defined in (17). Define a mapping $Q : (Y)_{\mathcal{U}} \to Y^{**}$ by setting for $(y_i)_{\mathcal{U}}$ and $g \in Y^*$

$$(Q(y_i)_{\mathcal{U}}, g) = \lim_{\mathcal{U}} (y_i, g).$$

For $x \in F$ we have, by (16), $Jx \in \mathcal{D}_0((S, F), \mathcal{U})$. Moreover, for $g \in Y^*$ we get

$$(Q(S)_{\mathcal{U}} Jx, g) = (Q(Sx)_{\mathcal{U}}, g) = (Sx, g).$$

Consequently,

$$Q(S)_{\mathcal{U}} Jx = K_Y Sx,$$

and hence,

$$PQ(S)_{\mathcal{U}} J|_F = S.$$

On the other hand, by (23) of Theorem 3.3, there are $\tilde{N} \in \mathcal{N}_n^{\mathrm{ad}}((X)_{\mathcal{U}})$ and $\tilde{\varphi} \in \Phi_n((Y)_{\mathcal{U}})$ such that

$$e\big((S)_{\mathcal{U}}, \tilde{\varphi} \circ \tilde{N}, \mathcal{D}_0((S, F), \mathcal{U}), (Y)_{\mathcal{U}}\big) \leq e_n(S, F, Y). \tag{44}$$

Now we put $N = \tilde{N} \circ J \in \mathcal{N}_n^{\mathrm{ad}}(X)$ (see (4) and (5)) and $\varphi = PQ \circ \tilde{\varphi} \in \Phi_n(Y)$. Then, since by (16), $J(F) \subset \mathcal{D}_0((S, F), \mathcal{U}), (Y)_{\mathcal{U}}\big)$ and $\|PQ\| = 1$, relation (6) of Lemma 2.1 together with (44) gives

$$\begin{aligned} e_n(S, F, Y) &\leq e(S, \varphi \circ N, F, Y) \\ &\leq e\big((S)_{\mathcal{U}}, \tilde{\varphi} \circ \tilde{N}, \mathcal{D}_0((S, F), \mathcal{U}), (Y)_{\mathcal{U}}\big) \leq e_n(S, F, Y). \end{aligned}$$

□

4 Local properties

The main theme of this section is the relation of the n-th minimal errors of an operator to those of its local, that is, finite dimensional parts (explained precisely in (47)). Throughout this section not only the original operator $S : F \to Y$ will play a role, but also its canonical extension $K_Y S : F \to Y^{**}$ to the bidual of Y. The first lemma, which is a consequence of the local reflexivity principle, Lemma 2.3, relates the n-th minimal errors of S to those of $K_Y S$.

Lemma 4.1. *Let X, Y be normed spaces and let $\emptyset \neq F \subset X$ and $S : F \to Y$ be arbitrary. If $S(F)$ is precompact or Y is 1-complemented in Y^{**}, then*

$$e_n(K_Y S, F, Y^{**}) = e_n(S, F, Y).$$

Proof. Since $\|K_Y\| = 1$, we always have $e_n(K_Y S, F, Y^{**}) \le e_n(S, F, Y)$. If P is a projection from Y^{**} to Y with $\|P\| = 1$, then

$$e_n(S, F, Y) \le \|P\| e_n(K_Y S, F, Y^{**}) = e_n(K_Y S, F, Y^{**}).$$

It remains to consider the case of precompact $S(F)$. Let $\delta > 0$ and let $N \in \mathcal{N}_n^{\mathrm{ad}}(X)$, $\tilde{\varphi} \in \Phi_n(Y^{**})$ be such that

$$\sup_{a \in N(F)} \sup_{x \in F, N(x)=a} \|S(x) - \tilde{\varphi}(a)\| \le e_n(K_Y S, F, Y^{**}) + \delta. \tag{45}$$

Fix any $a \in N(F)$. The set $\{S(x) : x \in F, N(x) = a\}$ is precompact in Y. Hence, there are $x_1, \dots, x_m \in F$ such that $N(x_k) = a$ $(k = 1, \dots, m)$ and

$$\sup_{x \in F, N(x)=a} \inf_{1 \le k \le m} \|S(x) - S(x_k)\| \le \delta.$$

By local reflexivity, see Lemma 2.3, there is a linear operator

$$T : \operatorname{span}\{S(x_1), \dots, S(x_m), \tilde{\varphi}(a)\} \to Y$$

with

$$\|T\| \le 1 + \delta \quad \text{and} \quad TS(x_k) = S(x_k) \quad (k = 1, \dots, m).$$

We put $\varphi(a) = T\tilde{\varphi}(a) \in Y$. Then

$$\begin{aligned}
\|S(x) - \varphi(a)\| &\le \max_{1 \le k \le m} \|S(x_k) - \varphi(a)\| + \delta \\
&= \max_{1 \le k \le m} \|TS(x_k) - T\tilde{\varphi}(a)\| + \delta \\
&\le (1 + \delta) \max_{1 \le k \le m} \|S(x_k) - \tilde{\varphi}(a)\| + \delta \\
&\le (1 + \delta) \sup_{x \in F, N(x)=a} \|S(x) - \tilde{\varphi}(a)\| + \delta. \qquad (46)
\end{aligned}$$

Extend φ defined so far on $N(F)$ in an arbitrary way to all of $\mathbb{R}^n$ so that $\varphi \in \Phi_n(Y)$. By (45) and (46)

$$e(S, \varphi \circ N, F, Y) \le (1 + \delta)(e_n(K_Y S, F, Y^{**}) + \delta) + \delta.$$

This shows that $e_n(S, F, Y) \le e_n(K_Y S, F, Y^{**})$ and concludes the proof.

□

Given a subspace $E \subset X$ and a closed subspace $G \subset Y$ we let $J_E : E \to X$ be the canonical embedding and $Q_G : Y \to Y/G$ the canonical quotient map. By $\mathrm{Dim}(X)$, respectively $\mathrm{Cod}(X)$, we denote the collection of all finite dimensional, respectively closed finite codimensional, subspaces of X. Furthermore, given a subset $\emptyset \neq F \subset X$ and a subspace $E \subset X$ with $F \cap E \neq \emptyset$, we let $J_{F\cap E} : F \cap E \to F$ be the embedding. Let $\mathrm{Dim}(F, X)$ be the set of all $E \in \mathrm{Dim}(X)$ with $F \cap E \neq \emptyset$.

Next we study the relation of n-th minimal errors of local parts of the operator S to the n-th minimal errors of S. By local (finite dimensional) parts we mean the operators

$$Q_G S J_{F\cap E} : F \cap E \xrightarrow{J_{F\cap E}} F \xrightarrow{S} Y \xrightarrow{Q_G} Y/G, \tag{47}$$

acting between finite dimensional spaces, where $E \in \mathrm{Dim}(F, X)$ and $G \in \mathrm{Cod}(Y)$). It turns out that, in general, the errors of the local parts are rather related to the errors of $K_Y S$ than to those of S.

Proposition 4.2. *Let X and Y be normed spaces, let $\emptyset \neq F \subset X$ be open and $S : F \to Y$ continuous. Then*

$$\begin{aligned} e_n(K_Y S, F, Y^{**}) &= \sup_{E\in\mathrm{Dim}(F,X), G\in\mathrm{Cod}(Y)} e_n(Q_G S J_{F\cap E}, F \cap E, Y/G) && (48)\\ &= \sup_{G\in\mathrm{Cod}(Y)} e_n(Q_G S, F, Y/G) && (49)\\ &= \sup_{E\in\mathrm{Dim}(F,X)} e_n(K_Y S J_{F\cap E}, F \cap E, Y^{**}). && (50) \end{aligned}$$

Moreover, if $S(F\cap E)$ is precompact for every $E \in \mathrm{Dim}(F, X)$, then we also have

$$e_n(K_Y S, F, Y^{**}) = \sup_{E\in\mathrm{Dim}(F,X)} e_n(S J_{F\cap E}, F \cap E, Y). \tag{51}$$

Proof. Let $I = \mathrm{Dim}(F, X) \times \mathrm{Cod}(Y)$ and let $\mathcal{F}$ be the filter of all sets $I_0 \subset I$ such that there exist $E_0 \in \mathrm{Dim}(F, X)$ and $G_0 \in \mathrm{Cod}(Y)$ with

$$I_0 = \{(E, G) : E_0 \subset E, G_0 \supset G\}.$$

Let $\mathcal{U}$ be an ultrafilter dominating $\mathcal{F}$. For the components of $i \in I$ we use the notation $i = (E_i, G_i)$. We can identify X with a subspace of $(E_i)_{\mathcal{U}}$ via the isometric embedding

$$J : X \to (E_i)_{\mathcal{U}}, \quad Jx = (x_i)_{\mathcal{U}},$$

where

$$x_i = \begin{cases} 0 & \text{if} \quad x \notin E_i \\ x & \text{if} \quad x \in E_i. \end{cases}$$

Furthermore, we define a mapping

$$Q : (Y/G_i)_{\mathcal{U}} \to Y^{**}$$

as follows. For $(z_i)_{\mathcal{U}} \in (Y/G_i)_{\mathcal{U}}$ we choose any family $(y_i) \in \ell_\infty(I, Y)$ with $Q_{G_i} y_i = z_i$ $(i \in I)$ and define $Q(z_i)_{\mathcal{U}} \in Y^{**}$ by setting for $g \in Y^*$

$$(Q(z_i)_{\mathcal{U}}, g) = \lim_{\mathcal{U}} g(y_i).$$

It is readily checked that this definition is correct and that $\|Q\| = 1$.

Let $x \in F$. First we prove that

$$Jx \in \mathcal{D}_0((Q_{G_i} S J_{F \cap E_i}, F \cap E_i), \mathcal{U}).$$

We have

$$\{i \in I :\, x_i \in F \cap E_i\} \in \mathcal{U},$$

which shows (11). Furthermore, for all $i \in I$ with $x \in E_i$ we have

$$Q_{G_i} S J_{F \cap E_i} x_i = Q_{G_i} S x, \tag{52}$$

hence

$$\|Q_{G_i} S J_{F \cap E_i} x_i\| \le \|Q_{G_i} S x\| \le \|Sx\|,$$

which implies (12). Let $(y_i) \in \ell_\infty(I, E_i)$ be such that $(y_i)_{\mathcal{U}} = Jx$. Then

$$\lim_{\mathcal{U}} \|y_i - x\| = 0$$

and, since F is open,

$$\{i \in I :\, y_i \in F \cap E_i\} = \{i \in I :\, y_i \in F\} \in \mathcal{U},$$

which shows (15). Moreover, by the continuity of S,

$$\lim_{\mathcal{U} | \{i \in I :\, y_i \in F \cap E_i\}} \|S(y_i) - S(x)\| = 0,$$

from which we infer

$$\lim_{\mathcal{U} | \{i \in I :\, x_i, y_i \in F \cap E_i\}} \|Q_{G_i} S J_{F \cap E_i} y_i - Q_{G_i} S J_{F \cap E_i} x_i\| = 0,$$

which is condition (13).

Next we prove that

$$Q(Q_{G_i} S J_{F \cap E_i})_{\mathcal{U}} Jx = K_Y S x. \tag{53}$$

It follows from (52) that

$$(Q_{G_i} S J_{F \cap E_i})_{\mathcal{U}} Jx = (Q_{G_i} S x)_{\mathcal{U}}. \tag{54}$$

By the definition of Q above, for any $g \in Y^*$

$$(Q(Q_{G_i} S x)_{\mathcal{U}}, g) = (Sx, g),$$

which together with (54) proves (53). So we have

$$K_Y S : F \xrightarrow{J} \mathcal{D}_0((Q_{G_i} S J_{F\cap E_i}, F\cap E_i), \mathcal{U}) \xrightarrow{(Q_{G_i} S J_{F\cap E_i})_{\mathcal{U}}} (Y/G_i)_{\mathcal{U}} \xrightarrow{Q} Y^{**}.$$

By Theorem 3.3,

$$\begin{aligned} & e_n(K_Y S, F, Y^{**}) \\ & \quad \leq \ e_n\big((Q_{G_i} S J_{F\cap E_i})_{\mathcal{U}}, \mathcal{D}_0((Q_{G_i} S J_{F\cap E_i}, F\cap E_i), \mathcal{U}), (Y/G_i)_{\mathcal{U}}\big) \\ & \quad \leq \ \lim_{\mathcal{U}} e_n(Q_{G_i} S J_{F\cap E_i}, F\cap E_i, Y/G_i). \end{aligned} \tag{55}$$

Now let $E \in \mathrm{Dim}(F, X)$, $G \in \mathrm{Cod}(Y)$. Since $Q_G = Q_G^{**} K_Y$, we have

$$e_n(Q_G S J_{F\cap E}, F\cap E, Y/G) \leq e_n(Q_G S, F, Y/G) \leq e_n(K_Y S, F, Y^{**}), \tag{56}$$

and similarly,

$$\begin{aligned} e_n(Q_G S J_{F\cap E}, F\cap E, Y/G) \ & \leq \ e_n(K_Y S J_{F\cap E}, F\cap E, Y^{**}) \\ & \leq \ e_n(K_Y S, F, Y^{**}). \end{aligned} \tag{57}$$

Combining (55–57) completes the proof of (48–50).

If $S(F\cap E)$ is precompact, then (51) follows from (50) and Lemma 4.1.

□

Using properties of Gelfand numbers, it was observed in [5] that for bounded linear $S \in L(X, Y)$

$$e_n(S, \mathcal{B}_X, Y) \leq 2 \sup_{E\in \mathrm{Dim}(F,X)} e_n(S|_E, \mathcal{B}_E, Y). \tag{58}$$

As a first consequence of Proposition 4.2 we get a generalization of (58) to the nonlinear situation.

Corollary 4.3. *Let $\emptyset \neq F \subset X$ be open and $S : F \to Y$ continuous. Then*

$$e_n(S, F, Y) \ \leq \ 2 \sup_{E\in \mathrm{Dim}(F,X), G\in \mathrm{Cod}(Y)} e_n(Q_G S J_{F\cap E}, F\cap E, Y/G) \tag{59}$$

$$e_n(S, F, Y) \ \leq \ 2 \sup_{G\in \mathrm{Cod}(Y)} e_n(Q_G S, F, Y/G) \tag{60}$$

$$e_n(S, F, Y) \ \leq \ 2 \sup_{E\in \mathrm{Dim}(F,X)} e_n(S J_{F\cap E}, F\cap E, Y). \tag{61}$$

Proof. Relations (59) and (60) follow from (8) of Lemma 2.1 and (48) and (49) of Proposition 4.2. Similarly, (61) follows from (8) and (50), taking into account that

$$e_n(K_Y S J_{F\cap E}, F\cap E, Y^{**}) \leq e_n(S J_{F\cap E}, F\cap E, Y).$$

□

The following corollary, which is a direct consequence of Proposition 4.2 and Lemma 4.1, shows that under certain restrictions the factor 2 in Corollary 4.3 can be removed.

Corollary 4.4. *Let $\emptyset \neq F \subset X$ be open and $S : F \to Y$ continuous. If $S(F)$ is precompact or Y is 1-complemented in Y^{**}, then*

$$\begin{aligned} e_n(S, F, Y) &= \sup_{E \in \mathrm{Dim}(F,X), G \in \mathrm{Cod}(Y)} e_n(Q_G S J_{F\cap E}, F \cap E, Y/G) \\ &= \sup_{G \in \mathrm{Cod}(Y)} e_n(Q_G S, F, Y/G) \\ &= \sup_{E \in \mathrm{Dim}(F,X)} e_n(S J_{F\cap E}, F \cap E, Y). \end{aligned} \tag{62}$$

Relation (62) confirms the 'at least' part of a conjecture made in [5], see relation (3) of that paper. Precisely, it was conjectured there that (62) holds if $S(F)$ is precompact.

Now we turn to an example which will show the limitations of Corollaries 4.3 and 4.4. Let $J_{1,0} : \ell_1 \to c_0$ be the identical embedding. Then $K_{c_0} J_{1,0} : \ell_1 \to c_0^{**} = \ell_\infty$ is the identical embedding of ℓ_1 into ℓ_∞. The following is inspired by Proposition 11.11.10 of [9].

Proposition 4.5. *For all $n \in \mathbb{N}_0$*

$$e_n(J_{1,0}, \mathcal{B}_{\ell_1}, c_0) = 1 \tag{63}$$

and for all $n \in \mathbb{N}$

$$e_n(K_{c_0} J_{1,0}, \mathcal{B}_{\ell_1}, \ell_\infty) = \frac{1}{2}. \tag{64}$$

Proof. Relation (64) is a direct consequence of (10) and [9], Propositions 11.11.10 and 11.5.3. The upper bound of (63) is obvious. To show the lower bound, let $N \in \mathcal{N}_n^{\mathrm{ad}}(\ell_1)$, $N = (L_1, \dots, L_n)$, $\varphi \in \Phi_n(c_0)$. We assume that N satisfies the conclusions of Lemma 3.1. Let $\mathcal{U}$ be a non-trivial ultrafilter on $\mathbb{N}$ and let $0 < \delta < 1$. Define

$$\begin{aligned} f_1 &= L_1, \quad f_1 = (f_{1,i})_{i=1}^\infty \in \ell_\infty \\ a_1 &= (1-\delta) \lim_{\mathcal{U}} f_{1,i} \\ f_2 &= L_2(\,\cdot\,, a_1), \quad f_2 = (f_{2,i})_{i=1}^\infty \in \ell_\infty \\ a_2 &= (1-\delta) \lim_{\mathcal{U}} f_{2,i} \\ \dots & \quad \dots \\ f_n &= L_n(\,\cdot\,, a_1, \dots, a_{n-1}), \quad f_n = (f_{n,i})_{i=1}^\infty \in \ell_\infty \\ a_n &= (1-\delta) \lim_{\mathcal{U}} f_{n,i} \end{aligned}$$

and $a = (a_1, \dots, a_n)$. Since N satisfies the conclusions of Lemma 3.1, the set $\{f_1, \dots, f_n\} \subset \ell_\infty$ is linearly independent. Using local reflexivity, Lemma 2.3, it follows that there exist $x_k = (x_{k,i})_{i=1}^\infty \in \ell_1$ $(1 \le k \le n)$ with $f_j(x_k) = \delta_{jk}$. For $i \in \mathbb{N}$ let e_i denote the i-th unit vector in ℓ_1 and define

$$y_i = (1-\delta)e_i + \sum_{k=1}^n (a_k - (1-\delta)f_{k,i})x_k.$$

Then for $1 \le j \le n$ and $i \in \mathbb{N}$ we have $f_j(y_i) = a_j$, hence

$$N(y_i) = a. \tag{65}$$

Moreover,

$$\lim_{\mathcal{U}} \|y_i\|_{\ell_1} = 1 - \delta. \tag{66}$$

Let $\varphi(a) = (\zeta_i)_{i=1}^\infty \in c_0$. Then we have

$$\lim_{\mathcal{U}} \|y_i - \varphi(a)\|_{c_0} \ge \lim_{\mathcal{U}} \left|1 - \delta - \zeta_i + \sum_{k=1}^n (a_k - (1-\delta)f_{k,i})x_{k,i}\right| = 1 - \delta. \tag{67}$$

By (65–67), there is a set $I_0 \in \mathcal{U}$ such that for $i \in I_0$,

$$N(y_i) = a, \quad \|y_i\|_{\ell_1} \le 1, \quad \|y_i - \varphi(a)\|_{c_0} \ge 1 - 2\delta,$$

and we conclude

$$e(J_{1,0}, \varphi \circ N, \mathcal{B}_{l_1}, c_0) \ge 1 - 2\delta.$$

Since N, φ, and δ were arbitrary, the lower bound of (63) follows. $\square$

Proposition 4.5 shows that without the assumptions on S or Y, Lemma 4.1 does not hold, in general. The next result, which is also a consequence of Proposition 4.5, will be formulated for the open ball because it serves as a counterexample to generalizations of Corollary 4.4. We note, however, that by (9) of Lemma 2.1, for all $S \in L(X, Y)$

$$e_n(S, \mathcal{B}_X^\circ, Y) = e_n(S, \mathcal{B}_X, Y). \tag{68}$$

Corollary 4.6. *We have*

$$e_n(J_{1,0}, \mathcal{B}_{\ell_1}^\circ, c_0) = 1, \tag{69}$$

and

$$\sup_{E \in \mathrm{Dim}(\ell_1), G \in \mathrm{Cod}(c_0)} e_n(Q_G J_{1,0} J_{\mathcal{B}_{\ell_1}^\circ \cap E}, \mathcal{B}_{\ell_1}^\circ \cap E, c_0/G)$$

$$= \sup_{G \in \mathrm{Cod}(c_0)} e_n(Q_G J_{1,0}, \mathcal{B}_{\ell_1}^\circ, c_0/G) \tag{70}$$

$$= \sup_{E \in \mathrm{Dim}(\ell_1)} e_n(J_{1,0} J_{\mathcal{B}_{\ell_1}^\circ \cap E}, \mathcal{B}_{\ell_1}^\circ \cap E, c_0) = \frac{1}{2}. \tag{71}$$

Proof. Relation (69) follows from (63) and (68). Similarly, relations (70–71) follow from (48), (49), (51), (64), and (68), where we note that, because J is a bounded linear operator, $J(\mathcal{B}^\circ_{\ell_1} \cap E)$ is precompact for all $E \in \mathrm{Dim}(\ell_1)$. □

Corollary 4.6 shows that the factor 2 in Corollary 4.3 is sharp and that, in general, without the assumptions on S or Y, Corollary 4.4 does not hold. This disproves the 'general version' of the already mentioned conjecture in [5], relation (3) (that is, the conjecture that (62) holds for all continuous operators).

5 Further results and comments

5.1 Another counterexample

Let us give an example along the same line as that in Proposition 4.5, which shows that relation (51) in Proposition 4.2 may fail without the assumption of precompactness of $S(F \cap E)$. Let X be any infinite dimensional normed space, let $F = \mathcal{B}^\circ_X$, and let $h : [0, 1) \to c_0$ be defined as follows. We put

$$h\left(1 - \frac{1}{i}\right) = e_i \quad (i \in \mathbb{N}),$$

where e_i is the i-th unit vector in c_0, and interpolate linearly within the intervals $[1-1/i, 1-1/(i+1)]$. Clearly, h is continuous on $[0, 1)$, $h(t) \geq 0$, and $\|h(t)\|_{c_0} \leq 1$ for all $t \in [0, 1)$. Now we define $S : \mathcal{B}^\circ_X \to c_0$ by setting $Sx = h(\|x\|)$ for $x \in \mathcal{B}^\circ_X$. Let $z_0 = (\frac{1}{2}, \frac{1}{2}, \dots) \in \ell_\infty$. Then $\|h(t) - z_0\|_{\ell_\infty} \leq 1/2$, so

$$e_0(K_{c_0} S, \mathcal{B}^\circ_X, \ell_\infty) = 1/2$$

(the lower bound follows from $\|e_i - e_{i+1}\|_{\ell_\infty} = 1$). Next we show that for any $n \in \mathbb{N}_0$ and any $E \subset X$ with $n + 1 \leq \dim E < \infty$

$$e_n(S J_{\mathcal{B}^\circ_X \cap E}, \mathcal{B}^\circ_X \cap E, c_0) = 1. \tag{72}$$

Indeed. the upper bound is obvious. To check the lower bound, we fix n and E and let $N \in \mathcal{N}^{\mathrm{ad}}_n(X)$, $N = (L_1, \dots, L_N)$ and $\varphi \in \Phi_n(c_0)$. Define $f_1, \dots, f_n \in X^*$ by

$$\begin{aligned} f_1 &= L_1 \\ f_k &= L_k(\,\cdot\,, 0, \dots, 0) \quad (2 \leq k \leq n). \end{aligned}$$

Let $x_0 \in E$ be any element with $\|x_0\| = 1$ and $f_k(x_0) = 0$ $(k = 1, \dots, n)$. For any $t \in [0, 1)$ we have $tx_0 \in \mathcal{B}^\circ_X \cap E$ and $N(tx_0) = 0$, hence

$$\begin{aligned} \sup_{x \in \mathcal{B}^\circ_X \cap E, N(x)=0} \|S(x) - \varphi(0)\|_{c_0} &\geq \sup_{t \in [0,1)} \|S(tx_0) - \varphi(0)\|_{c_0} \\ &\geq \sup_{i \in \mathbb{N}} \|e_i - \varphi(0)\|_{c_0} = 1, \end{aligned}$$

which implies (72).

5.2 More on the ultraproduct

Here we want to comment on the nonlinear ultraproduct construction and the relation of the two domains of definition given in Section 2. First of all, we introduce two concepts of the ultraproduct of a family of subsets. Let I be a nonempty set, $\mathcal{U}$ an ultrafilter on I, X_i normed spaces and $\emptyset \neq F_i \subset X_i$ arbitrary subsets ($i \in I$). Define $(F_i)_\mathcal{U} \subset (X_i)_\mathcal{U}$ to be the set of all $x \in (X_i)_\mathcal{U}$ such that *there exists* a family $(x_i) \in \ell_\infty(I, X_i)$ with $(x_i)_\mathcal{U} = x$ and $\{i \in I : x_i \in F_i\} \in \mathcal{U}$. Furthermore, define $[F_i]_\mathcal{U}$ as the the set of all $x \in (F_i)_\mathcal{U}$ such that *each* family $(x_i) \in \ell_\infty(I, \operatorname{span} F_i)$ with $(x_i)_\mathcal{U} = x$ satisfies $\{i \in I : x_i \in F_i\} \in \mathcal{U}$. By definition,

$$[F_i]_\mathcal{U} \subset (F_i)_\mathcal{U},$$

and if $F_i = X_i$ for all $i \in I$, then $[F_i]_\mathcal{U} = (F_i)_\mathcal{U} = (X_i)_\mathcal{U}$. Furthermore,

$$[\mathcal{B}_{X_i}]_\mathcal{U} = \mathcal{B}^\circ_{(X_i)_\mathcal{U}} \quad \text{and} \quad (\mathcal{B}_{X_i})_\mathcal{U} = \mathcal{B}_{(X_i)_\mathcal{U}}.$$

Let Y_i ($i \in I$) be normed spaces. As usual, we call a family of mappings $S_i : F_i \to Y_i$ uniformly equicontinuous, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for all i and all $x, y \in F_i$ with $\|x - y\| \leq \delta$ we have $\|S_i(x) - S_i(y)\| \leq \varepsilon$. The family is said to be uniformly bounded, if for each $c > 0$ there is a $C > 0$ such that for all $i \in I$ and for all $x \in F_i$ with $\|x\| \leq c$ we have $\|S_i(x)\| \leq C$.

It is easily checked that if (S_i) is uniformly equicontinuous and uniformly bounded, then

$$\mathcal{D}_0((S_i, F_i), \mathcal{U}) = [F_i]_\mathcal{U} \tag{73}$$
$$\mathcal{D}((S_i, F_i), \mathcal{U}) = (F_i)_\mathcal{U}. \tag{74}$$

In particular, if $F_i = \mathcal{B}_{X_i}$ for all $i \in I$, then

$$\begin{aligned} \mathcal{D}_0((S_i, \mathcal{B}_{X_i}), \mathcal{U}) &= \mathcal{B}^\circ_{(X_i)_\mathcal{U}} \qquad (75)\\ \mathcal{D}((S_i, \mathcal{B}_{X_i}), \mathcal{U}) &= \mathcal{B}_{(X_i)_\mathcal{U}}. \end{aligned}$$

In view of (73) and (74) let us make some more comments on $[F_i]_\mathcal{U}$ and $(F_i)_\mathcal{U}$. We have the following relation between them, which shows that both definitions are, in a sense, complementary:

$$\begin{aligned} [F_i]_\mathcal{U} \cap ((\operatorname{span} F_i) \setminus F_i)_\mathcal{U} &= \emptyset \\ [F_i]_\mathcal{U} \cup ((\operatorname{span} F_i) \setminus F_i)_\mathcal{U} &= (\operatorname{span} F_i)_\mathcal{U}. \end{aligned}$$

For the case of a countably incomplete ultrafilter $\mathcal{U}$ we can characterize $[F_i]_\mathcal{U}$ as follows.

Lemma 5.1. *If $\mathcal{U}$ is countably incomplete, then $[F_i]_\mathcal{U}$ consists of all $x \in (X_i)_\mathcal{U}$ such that there is a $\delta > 0$ and a family $(x_i) \in \ell_\infty(I, X_i)$ with $(x_i)_\mathcal{U} = x$ and*

$$\{i \in I : x_i + \delta \mathcal{B}_{\operatorname{span} F_i} \subset F_i\} \in \mathcal{U}. \tag{76}$$

Proof. Clearly, each $x \in (X_i)_{\mathcal{U}}$ which satisfies (76) belongs to $[F_i]_{\mathcal{U}}$. Now let $x \in [F_i]_{\mathcal{U}}$. We show that for each family $(x_i) \in \ell_\infty(I, \operatorname{span} F_i)$ with $(x_i)_{\mathcal{U}} = x$ there is a $\delta > 0$ such that (76) holds. For this purpose, assume the contrary, that is, there is a family $(x_i) \in \ell_\infty(I, \operatorname{span} F_i)$ such that $(x_i)_{\mathcal{U}} = x$ and for each $k \in \mathbb{N}$

$$J_k = \{i \in I : (x_i + k^{-1}\mathcal{B}_{\operatorname{span} F_i}) \setminus F_i \neq \emptyset\} \in \mathcal{U}. \tag{77}$$

We have $J_k \supset J_{k+1}$ $(k \in \mathbb{N})$. Let $(I_k)_{k=1}^\infty \subset \mathcal{U}$ be such that $I_1 \supset I_2 \supset \dots$ and $\cap_{k=1}^\infty I_k = \emptyset$. Then $I_k \cap J_k \in \mathcal{U}$ $(k \in \mathbb{N})$ and $\cap_{k=1}^\infty (I_k \cap J_k) = \emptyset$. By (77), for each $i \in (I_k \cap J_k) \setminus (I_{k+1} \cap J_{k+1})$ we can find a $y_i \in \operatorname{span} F_i$ with $y_i \notin F_i$ and $\|y_i - x_i\| \le k^{-1}$. This defines y_i for all $i \in I_1 \cap J_1$. For $i \notin I_1 \cap J_1$ we put $y_i = 0$. Then $(y_i) \in \ell_\infty(I, \operatorname{span} F_i)$, $(y_i)_{\mathcal{U}} = (x_i)_{\mathcal{U}}$, but $\{i \in I : y_i \in F_i\} \notin \mathcal{U}$, contradicting the definition of $[F_i]_{\mathcal{U}}$.

□

5.3 The linear case

In this section we only consider bounded linear operators between Banach spaces. For $S \in L(X,Y)$ we write $e_n(S)$ instead of $e_n(S, \mathcal{B}_X, Y)$. Following Pietsch [9], we say that a mapping, which assigns to each $S \in L(X,Y)$ and each $n \in \mathbb{N}_0$ a real number $s_n(S)$, is an s-function, if the following conditions (78-82) hold:

For Banach spaces X, X_1, Y, Y_1, operators $S, T \in L(X,Y)$, $U \in L(X_1, X)$, $V \in L(Y, Y_1)$, $n \in \mathbb{N}_0$

$$\begin{aligned} \|S\| &= s_0(S) \ge s_1(S) \ge \dots \ge 0 && (78)\\ s_n(S+T) &\le s_n(S) + \|T\| && (79)\\ s_n(VSU) &\le \|V\| s_n(S) \|U\|. && (80) \end{aligned}$$

If $\operatorname{rank}(S) \le n$ then

$$s_n(S) = 0. \tag{81}$$

If H is a Hilbert space with $\dim(H) \ge n+1$, then

$$s_n(I_H) = 1, \tag{82}$$

where I_H denotes the identity of H.

Corollary 5.2. *The n-th minimal errors e_n constitute an s-function.*

Proof. Relations (78), (79), and (81) are obvious consequences of the definition of the e_n, while (80) follows from (7) and (9) of Lemma 2.1. Relation (82) follows from Lemma 2.2 and the respective property of the Gelfand numbers.

□

An s-function is called ultrastable (see [9], 11.10.5), if for all sets I, ultrafilters $\mathcal{U}$ on I, families of Banach spaces X_i, Y_i, operators $S_i \in L(X_i, Y_i)$ $(i \in I)$ with $\lim_{\mathcal{U}} \|S_i\| < \infty$ and all $n \in \mathbb{N}_0$ we have

$$s_n((S_i)_{\mathcal{U}}) \le \lim_{\mathcal{U}} s_n(S_i).$$

Corollary 5.3. *The n-th minimal errors are ultrastable.*

Proof. For $S_i \in L(X_i, Y_i)$ with $\lim_{\mathcal{U}} \|S_i\| < \infty$ and $F_i = \mathcal{B}_{X_i}$ $(i \in I)$ we have by (75)

$$\mathcal{D}_0((S_i, \mathcal{B}_{X_i}), \mathcal{U}) = \mathcal{B}^{\circ}_{(X_i)_{\mathcal{U}}},$$

so by (68)

$$e_n((S_i)_{\mathcal{U}}, \mathcal{D}_0((S_i, \mathcal{B}_{X_i}), \mathcal{U}), (Y_i)_{\mathcal{U}}) = e_n((S_i)_{\mathcal{U}}).$$

Now the statement follows from Theorem 3.3. □

An s-function is called regular (see [9], 11.7.1), if for all Banach spaces X, Y, all operators $S \in L(X, Y)$ and all $n \in \mathbb{N}_0$

$$s_n(K_Y S) = s_n(S).$$

An s-function is called maximal (see [9], 11.10.1 and 11.10.2), if for all Banach spaces X, Y, all operators $S \in L(X, Y)$ and all $n \in \mathbb{N}_0$

$$e_n(S) = \sup_{E \in \mathrm{Dim}(X), G \in \mathrm{Cod}(Y)} e_n(Q_G S J_E).$$

Corollary 5.4. *The n-th minimal errors are neither regular nor maximal.*

Proof. This is a direct consequence of Proposition 4.5, Corollary 4.6 and relation (68). □

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