

Quantum Boolean Summation with Repetitions in the Worst-Average Setting [★]

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Summary. We study the quantum summation (**QS**) algorithm of Brassard, Høyer, Mosca and Tapp, see [1], which approximates the arithmetic mean of a Boolean function defined on N elements. We present sharp error bounds of the **QS** algorithm in the worst-average setting with the average performance measured in the L_q norm, $q \in [1, \infty]$.

We prove that the **QS** algorithm with M quantum queries, $M < N$, has the worst-average error bounds of the form $\Theta(\ln M/M)$ for $q = 1$, $\Theta(M^{-1/q})$ for $q \in (1, \infty)$, and is equal to 1 for $q = \infty$. We also discuss the asymptotic constants of these estimates.

We improve the error bounds by using the **QS** algorithm with repetitions. Using the number of repetitions which is independent of M and linearly dependent on q , we get the error bound of order M^{-1} for any $q \in [1, \infty)$. Since $\Omega(M^{-1})$ is a lower bound on the worst-average error of any quantum algorithm with M queries, the **QS** algorithm with repetitions is optimal in the worst-average setting.

1.1 Introduction

The quantum summation (**QS**) algorithm of Brassard, Høyer, Mosca and Tapp computes an approximation to the arithmetic mean of the values of a Boolean function defined on a set of $N = 2^n$ elements. An overview of the **QS** algorithm and its basic properties is presented in the first two sections of [4]. In Section 1.2 we remind the reader of the facts concerning the **QS** algorithm that are needed in this paper.

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The **QS** algorithm enjoys many optimality properties and has many applications. It is used for the summation of real numbers which in turn is an essential part for many continuous problems such as multivariate and path integration, and multivariate approximation. The knowledge of the complexity of the quantum summation problem allows us to determine the quantum complexity of many continuous problems, such as those mentioned above, see [6] and a recent survey [3].

The **QS** algorithm has been studied in the two error settings so far:

- worst-probabilistic in [1, 4],
- average-probabilistic in [4].

These settings are defined by taking the worst case/average performance with respect to all Boolean functions and the probabilistic performance with respect to outcomes of the **QS** algorithm.

It turns out that the **QS** algorithm is optimal in these two settings. The corresponding lower bounds for the Boolean summation problem were shown in [5] for the worst-probabilistic setting, and in [7] for the average-probabilistic setting. In particular, we know that the **QS** algorithm with M quantum queries, $M < N$, has the error bound of order M^{-1} in the worst-probabilistic setting.

In this paper we study the *worst-average* setting. In this setting, we take the worst case performance over all Boolean functions and the average performance over all outcomes of the **QS** algorithm. The average performance is measured in the L_q norm, $q \in [1, \infty]$. This setting is analogous to the randomized (Monte Carlo) setting used for algorithms on a classical computer. The worst-average setting also seems to be quite natural for the analysis of quantum algorithms.

As we shall see, the results depend on the choice of q . Obviously, for larger q , the effect of the average behavior becomes less significant. In fact, the limiting case, $q = \infty$, leads to the deterministic case (modulo sets of measure zero). Not surprisingly, for $q = \infty$, the results are negative.

In what follows we indicate error bounds for large M . Since we always assume that $M < N$, this means that for M tending to infinity we also let N tend to infinity. To make error bounds independent of N , we take the supremum over $N > M$ in the corresponding definitions of the errors. When we speak about the sharpness of error bounds, we usually take a large M and select a still larger N and a Boolean function for which the presented error bound is sharp.

The worst-average error $e_q^{\text{wor-avg}}(M)$ of the **QS** algorithm with M quantum queries satisfies:

- For $q = 1$, we have $e_1^{\text{wor-avg}}(M) = \Theta\left(\frac{\ln M}{M}\right)$. Furthermore, the asymptotic constant is $2/\pi$ for $M - 2$ divisible by 4.

- For $q \in (1, \infty)$, we have $e_q^{\text{wor-avg}}(M) = \Theta\left(\frac{1}{M^{1/q}}\right)$. Furthermore, the asymptotic constant is approximately $(\int_0^\pi \sin^{q-2}(x)dx/\pi)^{1/q}$ for $M - 2$ divisible by 4 and q close to 1.
- For $q = \infty$, we have $e_\infty^{\text{wor-avg}}(M) = 1$.

The error bounds of the **QS** algorithm are improved by the use of repetitions. Namely, we repeat the **QS** algorithm $2n + 1$ times and take the median of the outputs obtained as the final output. This procedure boosts the success probability of the approximation at the expense of the number of quantum queries. We show that with n independent of M and linearly dependent on q , we decrease the **QS** algorithm error to be of order M^{-1} . Hence, the use of repetitions is particularly essential for large q since we change the error bound $O(M^{-1/q})$ without repetitions to the error bound $O(M^{-1})$ with repetitions. The constant in the last big O notation is absolute and does not depend on q and M .

The error bound of order M^{-1} is optimal. This follows from the use of, for instance, Chebyshev's inequality and the fact that the lower bound $\Omega(M^{-1})$ is sharp in the worst-probabilistic setting, see also [7]. Hence, the **QS** algorithm with repetitions is optimal in the worst-average setting.

1.2 Quantum Summation Algorithm

The quantum summation **QS** algorithm of Brassard, Høyer, Mosca and Tapp, see [1], approximates the mean

$$a_f = \frac{1}{N} \sum_{i=0}^{N-1} f(i)$$

of a Boolean function $f : \{0, 1, \dots, N-1\} \rightarrow \{0, 1\}$. Without loss of generality we assume that N is a power of two.

The **QS** algorithm uses $M - 1$ quantum queries. The only interesting case is when M is much smaller than N . The **QS** algorithm returns an index $j \in \{0, 1, \dots, M - 1\}$ with probability

$$p_f(j) = \frac{\sin^2(M\theta_{a_f})}{2M^2} \left(\sin^{-2}\left(\frac{\pi(j - \sigma_{a_f})}{M}\right) + \sin^{-2}\left(\frac{\pi(j + \sigma_{a_f})}{M}\right) \right),$$

see [4] for the detailed analysis of the **QS** algorithm. Here

$$\theta_{a_f} = \arcsin \sqrt{a_f} \quad \text{and} \quad \sigma_{a_f} = \frac{M}{\pi} \theta_{a_f}.$$

We will also be using

$$s_{a_f} = \min \{ \lceil \sigma_{a_f} \rceil - \sigma_{a_f}, \sigma_{a_f} - \lfloor \sigma_{a_f} \rfloor \}.$$

Clearly, $s_{a_f} \in [0, \frac{1}{2}]$ and $s_{a_f} = 0$ iff σ_{a_f} is an integer. We shall usually drop the subscript f and denote $\theta_a = \theta_{a_f}$, $\sigma_a = \sigma_{a_f}$, $s_a = s_{a_f}$ when f is clear from the context.

Knowing the index j , we compute the output

$$\bar{a}_f(j) = \sin^2\left(\frac{\pi j}{M}\right)$$

on a classical computer. The error is then given by

$$|a_f - \bar{a}_f(j)| = \left| \sin\left(\frac{\pi(j - \sigma_{a_f})}{M}\right) \sin\left(\frac{\pi(j + \sigma_{a_f})}{M}\right) \right|. \quad (1.1)$$

As in [4], we let $\mu(\cdot, f)$ denote the measure on the set of all possible outcomes of the **QS** algorithm which is defined as

$$\mu(A, f) = \sum_{j \in A} p_f(j) \quad \forall A \subset \{0, 1, \dots, M-1\}.$$

Let \mathcal{A}_M denote the set of all possible outputs of the **QS** algorithm with $M-1$ queries, i.e.,

$$\mathcal{A}_M = \left\{ \sin^2\left(\frac{\pi j}{M}\right) : j = 0, 1, \dots, M-1 \right\}.$$

Let

$$\rho_f(\alpha) = \mu\left(\left\{ j \in \{0, 1, \dots, M-1\} : \sin^2\left(\frac{\pi j}{M}\right) = \alpha \right\}, f\right) \quad \forall \alpha \in \mathcal{A}_M,$$

denote the probability of the output α . Note that $\alpha = \sin^2(\pi j/M) = \sin^2(\pi(M-j)/M)$. Hence if $j \neq 0$ and $j \neq M/2$ then $\rho_f(\alpha) = p_f(j) + p_f(M-j)$.

In what follows we let \mathbb{B}_N denote the set of all Boolean functions defined on $\{0, 1, \dots, N-1\}$.

1.3 Performance Analysis

The error of the **QS** algorithm in the worst-probabilistic and average-probabilistic settings has been analyzed in [1, 4]. In this paper we analyze the error of the **QS** algorithm in the worst-average setting. This corresponds to the worst case performance with respect to all Boolean functions from \mathbb{B}_N and the average performance with respect to all outcomes. This average performance is measured by the expectation in the L_q norm, $q \in [1, \infty]$, with respect to the probability measure of all outcomes provided by the **QS** algorithm. As mentioned before, we make the worst-average error independent of N by taking the supremum over $N > M$. That is, the worst-average error is defined as:

- for $q \in [1, \infty)$,

$$e_q^{\text{wor-avg}}(M) = \sup_{N > M} \max_{f \in \mathbb{B}_N} \left(\sum_{j=0}^{M-1} p_f(j) |a_f - \bar{a}_f(j)|^q \right)^{1/q},$$

- for $q = \infty$,

$$e_\infty^{\text{wor-avg}}(M) = \sup_{N > M} \max_{f \in \mathbb{B}_N} \max_{j: p_f(j) > 0} |a_f - \bar{a}_f(j)|.$$

It is easy to check that for $q = \infty$, the **QS** algorithm behaves badly. Indeed, if M is odd, we can take f with all values one, and then $a_f = 1$, $p_f(0) = 1/M^2$ and $\bar{a}_f(0) = 0$. Hence $e_\infty^{\text{wor-avg}}(M) = 1$. If M is even, we take f with only one value equal to 1, and then $a_f = 1/N$, $p_f(M/2) > 0$ and $\bar{a}_f(M/2) = 1$. Hence, $|a_f - \bar{a}_f(M/2)| = 1 - 1/N$ and $e_\infty^{\text{wor-avg}}(M) = 1$.

That is why in the rest of the paper we consider $q \in [1, \infty)$. As we shall see the cases $q > 1$ and $q = 1$ will require a different analysis and lead to quite different results.

1.3.1 Local Average Error

We analyze the local average error for a fixed function $f \in \mathbb{B}_N$ for $1 \leq q < \infty$,

$$e_q^{\text{avg}}(f, M) = \left(\sum_{j=0}^{M-1} p_f(j) |a_f - \bar{a}_f(j)|^q \right)^{1/q} = \left(\sum_{\alpha \in \mathcal{A}_M} \rho_f(\alpha) |a_f - \alpha|^q \right)^{1/q}. \quad (1.2)$$

We first analyze the case $q > 1$.

Theorem 1. *Let $q \in (1, \infty)$. Denote $a = a_f$. If $\sigma_a \in \mathbb{Z}$ then $e_q^{\text{avg}}(f, M) = 0$. If $\sigma_a \notin \mathbb{Z}$ then*

$$\left| e_q^{\text{avg}}(f, M)^q - \frac{\sin^2(\pi s_a)}{M\pi} \int_{\pi \bar{s}_a/M}^{\pi - \pi \underline{s}_a/M} \sin(x)^{q-2} |\sin(x + 2\theta_a)|^q dx \right| \leq (1 + 2(1 - \delta_{q,2})) \frac{\pi^{q-1} \sin(\pi s_a)}{M^q} + \frac{\sin^2(\pi s_a)}{M^2} \left(2(1 - \delta_{q,2}) + q \int_0^\pi \sin^{q-2}(x) dx \right), \quad (1.3)$$

with $\underline{s}_a = \lfloor \sigma_a \rfloor - \sigma_a$ and $\bar{s}_a = \sigma_a - \lceil \sigma_a \rceil$.⁴

⁴ Note that the last integral is finite. It is obvious for $q \geq 2$. For $q \in (1, 2)$, the only singularities are at the boundary points and are of the form x^{q-2} for x approaching 0. The function x^{q-2} is integrable since $q > 1$.

Proof. If $\sigma_a \in \mathbb{Z}$ then it is shown in [4] that there exists $\alpha \in \mathcal{A}_M$ such that $\alpha = a_f$ and $\rho_f(\alpha) = \delta_{\alpha, a_f}$ for all $\alpha \in \mathcal{A}_M$. Then $e_q^{\text{avg}}(f, M) = 0$ as claimed.

Assume that $\sigma_a \notin \mathbb{Z}$. Using the form of $p_f(j)$ from Section 1.2, we rewrite (1.2) as

$$(e_q^{\text{avg}}(f, M))^q = \sum_{j=0}^{M-1} \frac{\sin^2(M\theta_a)}{2M^2} \left(\left| \sin\left(\frac{\pi(j-\sigma_a)}{M}\right) \right|^{q-2} \left| \sin\left(\frac{\pi(j+\sigma_a)}{M}\right) \right|^q + \left| \sin\left(\frac{\pi(j+\sigma_a)}{M}\right) \right|^{q-2} \left| \sin\left(\frac{\pi(j-\sigma_a)}{M}\right) \right|^q \right).$$

We have

$$\begin{aligned} \sum_{j=0}^{M-1} \left| \sin\left(\frac{\pi(j+\sigma_a)}{M}\right) \right|^{q-2} \left| \sin\left(\frac{\pi(j-\sigma_a)}{M}\right) \right|^q \\ = \sum_{j=1}^M \left| \sin\left(\frac{\pi(M-j+\sigma_a)}{M}\right) \right|^{q-2} \left| \sin\left(\frac{\pi(M-j-\sigma_a)}{M}\right) \right|^q. \end{aligned}$$

Using the π -periodicity of $|\sin x|$, we see that the last sum is equal to

$$\begin{aligned} \sum_{j=1}^M \left| \sin\left(\frac{\pi(j-\sigma_a)}{M}\right) \right|^{q-2} \left| \sin\left(\frac{\pi(j+\sigma_a)}{M}\right) \right|^q \\ = \sum_{j=0}^{M-1} \left| \sin\left(\frac{\pi(j-\sigma_a)}{M}\right) \right|^{q-2} \left| \sin\left(\frac{\pi(j+\sigma_a)}{M}\right) \right|^q. \end{aligned}$$

Therefore

$$e_q^{\text{avg}}(f, M)^q = \frac{\sin^2(M\theta_a)}{M^2} S_{M,q} \tag{1.4}$$

with

$$\begin{aligned} S_{M,q} &= \sum_{j=0}^{M-1} \left| \sin\left(\frac{\pi(j-\sigma_a)}{M}\right) \right|^{q-2} \left| \sin\left(\frac{\pi(j+\sigma_a)}{M}\right) \right|^q \\ &= \sum_{j=0}^{M-1} \left| \sin\left(\frac{\pi j}{M} - \theta_a\right) \right|^{q-2} \left| \sin\left(\frac{\pi j}{M} + \theta_a\right) \right|^q. \end{aligned}$$

We split $S_{M,q}$ as

$$S'_{M,q} = S_{M,q} - \left| \sin\left(\frac{\pi \lfloor \sigma_a \rfloor}{M} - \theta_a\right) \right|^{q-2} \left| \sin\left(\frac{\pi \lfloor \sigma_a \rfloor}{M} + \theta_a\right) \right|^q.$$

Observe that $\frac{\pi}{M}S'_{M,q}$ is the rectangle formula for approximating the integral

$$\int_{[0,\pi] \setminus [\pi \lfloor \sigma_a \rfloor / M, \pi \lceil \sigma_a \rceil / M]} |\sin(x - \theta_a)|^{q-2} |\sin(x + \theta_a)|^q dx.$$

The error of the rectangle quadrature for $k \in \mathbb{N}$ and an absolutely continuous function $f : [a, b] \rightarrow \mathbb{R}$ whose first derivative belongs to $L_1([a, b])$ satisfies

$$\left| \int_a^b f(x) dx - \frac{b-a}{k} \sum_{j=0}^{k-1} f\left(a + j \frac{b-a}{k}\right) \right| \leq \frac{b-a}{k} \int_a^b |f'(x)| dx. \quad (1.5)$$

Thus defining $h(x) = |\sin(x - \theta_a)|^{q-2} |\sin(x + \theta_a)|^q$ and $D_a = [0, \pi] \setminus [\pi \lfloor \sigma_a \rfloor / M, \pi \lceil \sigma_a \rceil / M]$ and using the error formula above for the subintervals $[0, \pi \lfloor \sigma_a \rfloor / M]$ and $(\pi \lceil \sigma_a \rceil / M, \pi]$, we get

$$\left| \frac{\pi}{M} S'_{M,q} - \int_{D_a} h(x) dx \right| \leq \frac{\pi}{M} \int_{D_a} |h'(x)| dx.$$

Define $H(x) = h(x + \theta_a) = |\sin(x)|^{q-2} |\sin(x + 2\theta_a)|^q$ and $\Delta_a = [-\theta_a, \pi - \theta_a] \setminus [\pi(\lfloor \sigma_a \rfloor - \sigma_a) / M, \pi(\sigma_a - \lceil \sigma_a \rceil) / M]$. We have

$$\int_{D_a} h(x) dx = \int_{\Delta_a} H(x) dx, \quad \int_{D_a} |h'(x)| dx = \int_{\Delta_a} |H'(x)| dx.$$

and by the π -periodicity of the integrand H we have

$$\begin{aligned} \int_{\Delta_a} H(x) dx &= \int_{-\theta_a}^{-\pi \underline{\sigma}_a / M} H(x) dx + \int_{\pi \bar{\sigma}_a / M}^{\pi - \theta_a} H(x) dx \\ &= \int_{\pi - \theta_a}^{\pi - \pi \underline{\sigma}_a / M} H(x) dx + \int_{\pi \bar{\sigma}_a / M}^{\pi - \theta_a} H(x) dx = \int_{\pi \bar{\sigma}_a / M}^{\pi - \pi \underline{\sigma}_a / M} H(x) dx. \end{aligned}$$

Analogously,

$$\int_{\Delta_a} |H'(x)| dx = \int_{\pi \bar{\sigma}_a / M}^{\pi - \pi \underline{\sigma}_a / M} |H'(x)| dx.$$

For $x \in [\pi \bar{\sigma}_a / M, \pi - \pi \underline{\sigma}_a / M]$ the sine is positive and

$$|H'(x)| \leq |q-2| \sin^{q-3}(x) |\cos(x)| + q \sin^{q-2}(x).$$

It is easy to check that for $q \neq 2$ we have

$$\begin{aligned} &\int_{\pi \bar{\sigma}_a / M}^{\pi - \pi \underline{\sigma}_a / M} |q-2| \sin^{q-3}(x) |\cos(x)| dx \\ &= |q-2| \left(\int_{\pi \bar{\sigma}_a / M}^{\pi/2} \sin^{q-3}(x) d \sin(x) - \int_{\pi/2}^{\pi - \pi \underline{\sigma}_a / M} \sin^{q-3}(x) d \sin(x) \right) \\ &= \frac{|q-2|}{q-2} \left(2 - \sin^{q-2} \left(\frac{\pi \bar{\sigma}_a}{M} \right) - \sin^{q-2} \left(\pi - \frac{\pi \underline{\sigma}_a}{M} \right) \right). \end{aligned}$$

From this we get

$$\int_{\pi\bar{s}_a/M}^{\pi-\pi\underline{s}_a/M} |H'(x)| dx \leq (1 - \delta_{q,2}) \left(2 + \sin^{q-2} \left(\frac{\pi\bar{s}_a}{M} \right) + \sin^{q-2} \left(\frac{\pi\underline{s}_a}{M} \right) \right) + q \int_0^\pi \sin^{q-2}(x) dx.$$

We then finally get

$$\left| \frac{\pi}{M} S_{M,q} - \int_{\pi\bar{s}_a/M}^{\pi-\pi\underline{s}_a/M} H(x) dx \right| \leq \frac{\pi}{M} \left((1 - \delta_{q,2}) \left(2 + \sin^{q-2} \left(\frac{\pi\bar{s}_a}{M} \right) + \sin^{q-2} \left(\frac{\pi\underline{s}_a}{M} \right) \right) + \sin^{q-2} \left(\frac{\pi\underline{s}_a}{M} \right) + \sin^{q-2} \left(\frac{\pi\bar{s}_a}{M} \right) + q \int_0^\pi \sin^{q-2}(x) dx \right).$$

Observe also that

$$\sin(\pi s_a) = \sin(\pi \underline{s}_a) = \sin(\pi \bar{s}_a).$$

Since $\sin(x)/[M \sin(x/M)] \leq 1$ for $x \in (0, \pi]$, we get

$$\begin{aligned} & \left| \frac{\pi \sin(\pi s_a)}{M} S_{M,q} - \sin(\pi s_a) \int_{\pi\bar{s}_a/M}^{\pi-\pi\underline{s}_a/M} H(x) dx \right| \\ & \leq \pi(1 - \delta_{q,2}) \left(\sin^{q-1} \left(\frac{\pi\bar{s}_a}{M} \right) + \sin^{q-1} \left(\frac{\pi\underline{s}_a}{M} \right) \right) + \pi \sin^{q-1} \left(\frac{\pi\underline{s}_a}{M} \right) \\ & \quad + \frac{\pi \sin(\pi s_a)}{M} \left(2(1 - \delta_{q,2}) + q \int_0^\pi \sin^{q-2}(x) dx \right). \end{aligned}$$

Using $\sin(\pi\bar{s}_a/M) \leq \pi/M$ we obtain

$$\begin{aligned} & \left| \frac{\pi \sin(\pi s_a)}{M} S_{M,q} - \sin(\pi s_a) \int_{\pi\bar{s}_a/M}^{\pi-\pi\underline{s}_a/M} H(x) dx \right| \\ & \leq (1 + 2(1 - \delta_{q,2})) \frac{\pi^q}{M^{q-1}} + \frac{\pi \sin(\pi s_a)}{M} \left(2(1 - \delta_{q,2}) + q \int_0^\pi \sin^{q-2}(x) dx \right). \end{aligned}$$

Finally, since $\sin^2(M\theta_a) = \sin^2(\pi s_a)$, we complete the proof by using the estimate of $S_{M,q}$ in (1.4). \square

Theorem 1 implies the following corollary.

Corollary 1. *Let $q \in (1, \infty)$. If $\sigma_a \in \mathbb{Z}$ then $e_q^{\text{avg}}(f, M) = 0$. If $\sigma_a \notin \mathbb{Z}$ then*

$$e_q^{\text{avg}}(f, M) = \frac{1}{M^{1/q}} \left[\frac{\sin^2(\pi s_a)}{\pi} \left(\int_0^\pi \sin^{q-2}(x) |\sin(x + 2\theta_a)|^q dx + O\left(\frac{\sin(\pi s_a)}{M^{\min(1, q-1)}}\right) \right) \right]^{1/q}, \quad (1.6)$$

with $s_a \in (0, \frac{1}{2}]$, and the factor in the big O notation is independent of f from \mathbb{B}_N , and also independent of N .

We now consider the case $q = 1$ and present estimates of $e_1^{\text{avg}}(f, M)$ in the following lemma.

Lemma 1. *Let $a = a_f$. If $\sigma_a \in \mathbb{Z}$ then $e_1^{\text{avg}}(f, M) = 0$. If $\sigma_a \notin \mathbb{Z}$ then*

$$\left| e_1^{\text{avg}}(f, M) - \frac{\sin^2(\pi s_a) \sin(2\theta_a)}{M} \Sigma_{M, a} \right| \leq \frac{\sin^2(\pi s_a)}{M} |\cos(2\theta_a)|, \quad (1.7)$$

where $s_a \in (0, \frac{1}{2}]$, and

$$\Sigma_{M, a} = \frac{1}{M} \sum_{j=0}^{M-1} \left| \cot\left(\frac{\pi(j + s_a)}{M}\right) \right|.$$

Proof. The case $\sigma_a \in \mathbb{Z}$ can be proved as in Theorem 1. Assume that $\sigma_a \notin \mathbb{Z}$. Using the form of $p_f(j)$ from Section 1.2, we have

$$e_1^{\text{avg}}(f, M) = \sum_{j=0}^{M-1} \frac{\sin^2(M\theta_a)}{2M^2} \left(\left| \frac{\sin(\pi(j + \sigma_a)/M)}{\sin(\pi(j - \sigma_a)/M)} \right| + \left| \frac{\sin(\pi(j - \sigma_a)/M)}{\sin(\pi(j + \sigma_a)/M)} \right| \right)$$

As in the proof of Theorem 1 we conclude that

$$e_1^{\text{avg}}(f, M) = \frac{\sin^2(M\theta_a)}{M^2} S_{M, 1},$$

where

$$S_{M, 1} = \sum_{j=0}^{M-1} \left| \frac{\sin(\pi(j + \sigma_a)/M)}{\sin(\pi(j - \sigma_a)/M)} \right| = \sum_{j=0}^{M-1} \left| \frac{\sin(\pi(j - \lceil \sigma_a \rceil + \bar{\sigma}_a)/M + 2\theta_a)}{\sin(\pi(j - \lceil \sigma_a \rceil + \bar{\sigma}_a)/M)} \right|,$$

with $\bar{\sigma}_a = \lceil \sigma_a \rceil - \sigma_a$. Changing the index j in the second sum to $j - \lceil \sigma_a \rceil$, and using periodicity of the sine, we get

$$S_{M, 1} = \sum_{j=0}^{M-1} \left| \frac{\sin(\pi(j + \bar{\sigma}_a)/M + 2\theta_a)}{\sin(\pi(j + \bar{\sigma}_a)/M)} \right|$$

and consequently

$$S_{M,1} = \sum_{j=0}^{M-1} \left| \cos(2\theta_a) + \sin(2\theta_a) \cot\left(\frac{\pi(j + \bar{s}_a)}{M}\right) \right|.$$

Using the triangle inequality twice, we obtain

$$\left| S_{M,1} - \sin(2\theta_a) \sum_{j=0}^{M-1} \left| \cot\left(\frac{\pi(j + \bar{s}_a)}{M}\right) \right| \right| \leq M |\cos(2\theta_a)|.$$

Let $\underline{s}_a = \sigma_a - \lfloor \sigma_a \rfloor$. Observe that $\underline{s}_a = 1 - \bar{s}_a$. Since the cotangent is π -periodic and the function $|\cot(\pi(\cdot)/M)|$ is even, we get

$$\sum_{j=0}^{M-1} \left| \cot\left(\frac{\pi(j + \bar{s}_a)}{M}\right) \right| = \sum_{j=0}^{M-1} \left| \cot\left(\frac{\pi(j + \underline{s}_a)}{M}\right) \right| = M \Sigma_{M,a}.$$

This and

$$\sin^2(M\theta_a) = \sin^2(\pi\sigma_a) = \sin^2(\pi s_a)$$

yield (1.7) as claimed. \square

From Lemma 1 we see that the sum $\Sigma_{M,a}$ is the most important part of the local average error $e_1^{\text{avg}}(M, f)$. We now estimate $\Sigma_{M,a}$.

Lemma 2. *Assume that $\sigma_a \notin \mathbb{Z}$ and $M \geq 3$. Then*

$$\left| \Sigma_{M,a} - \frac{1}{M} \cot\left(\frac{\pi s_a}{M}\right) - \frac{1}{M} \left| \cot\left(\frac{\pi(M-1+s_a)}{M}\right) \right| - \frac{1}{\pi} \int_{\pi(1+s_a)/M}^{\pi(M-1+s_a)/M} |\cot x| dx \right| \leq \frac{1}{\pi M} \int_{\pi(1+s_a)/M}^{\pi(M-1+s_a)/M} \frac{1}{\sin^2 x} dx. \quad (1.8)$$

Proof. This can be shown by applying the error formula for rectangle quadratures (1.5). Note that $\pi \Sigma_{M,a} - \frac{\pi}{M} \cot(\pi s_a/M) - \frac{\pi}{M} \left| \cot(\pi(M-1+s_a)/M) \right|$ is the rectangle quadrature for the integral $\int_{\pi(1+s_a)/M}^{\pi(M-1+s_a)/M} |\cot x| dx$ with $k = M-2 \geq 1$. We then obtain (1.8) by using (1.5). \square

We now present the final estimate on the local average error $e_1^{\text{avg}}(f, M)$.

Theorem 2. *Assume that $f \in \mathbb{B}_N$ and $a = a_f$. For $M \geq 3$, the average error of the **QS** algorithm for the function f satisfies*

$$\left| e_1^{\text{avg}}(f, M) - \frac{2 \sin^2(\pi s_a) \sin(2\theta_a)}{\pi} \frac{\ln M}{M} \right| \leq \frac{3\pi + 2 + \ln(\pi^2)}{M\pi} \sin(\pi s_a). \quad (1.9)$$

Proof. For $s_a \in \mathbb{Z}$ we have $s_a = 0$ and (1.9) holds since $e_1^{\text{avg}}(f, M) = 0$ by [4]. Assume that $s_a \notin \mathbb{Z}$. From Lemmas 1 and 2 we have

$$\begin{aligned} & \left| e_1^{\text{avg}}(f, M) - \frac{\sin^2(\pi s_a) \sin(2\theta_a)}{\pi M} \int_{\pi(1+s_a)/M}^{\pi(M-1+s_a)/M} |\cot x| dx \right| \\ & \leq \frac{\sin^2(\pi s_a)}{M} \left[\frac{\sin(2\theta_a)}{M} \left(\cot\left(\frac{\pi s_a}{M}\right) + \left| \cot\left(\frac{\pi(M-1+s_a)}{M}\right) \right| \right) \right. \\ & \quad \left. + \frac{1}{\pi} \int_{\pi(1+s_a)/M}^{\pi(M-1+s_a)/M} \frac{1}{\sin^2 x} dx \right] + |\cos(2\theta_a)|. \end{aligned}$$

Observe that

$$\begin{aligned} & \int_{\pi(1+s_a)/M}^{\pi(M-1+s_a)/M} |\cot x| dx = \ln \left(\sin^{-1} \left(\frac{\pi(1+s_a)}{M} \right) \sin^{-1} \left(\frac{\pi(1-s_a)}{M} \right) \right), \\ & \left| \cot \left(\frac{\pi(M-1+s_a)}{M} \right) \right| = \cot \left(\frac{\pi(1-s_a)}{M} \right) \leq \cot \left(\frac{\pi s_a}{M} \right), \\ & \int_{\pi(1+s_a)/M}^{\pi(M-1+s_a)/M} \frac{1}{\sin^2 x} dx = \cot \left(\frac{\pi(1-s_a)}{M} \right) + \cot \left(\frac{\pi(1+s_a)}{M} \right) \\ & \leq 2 \cot \left(\frac{\pi s_a}{M} \right). \end{aligned}$$

The four formulas above yield

$$\begin{aligned} & \left| e_1^{\text{avg}}(f, M) - \frac{\sin^2(\pi s_a) \sin(2\theta_a)}{\pi M} \right. \\ & \quad \times \ln \left(\sin^{-1} \left(\frac{\pi(1+s_a)}{M} \right) \sin^{-1} \left(\frac{\pi(1-s_a)}{M} \right) \right) \left. \right| \\ & \leq \frac{\sin^2(\pi s_a)}{M} \left(\frac{(2+2/\pi) \sin(2\theta_a)}{M} \cot \left(\frac{\pi s_a}{M} \right) + |\cos(2\theta_a)| \right). \end{aligned}$$

Observe that $\sin(\pi s_a)/[M \sin(\pi s_a/M)] \leq 1$ since $s_a \in (0, \frac{1}{2}]$. This and the obvious estimates of sine and cosine yield

$$\begin{aligned} & \left| e_1^{\text{avg}}(f, M) - \frac{\sin^2(\pi s_a) \sin(2\theta_a)}{\pi M} \right. \\ & \quad \times \ln \left(\sin^{-1} \left(\frac{\pi(1+s_a)}{M} \right) \sin^{-1} \left(\frac{\pi(1-s_a)}{M} \right) \right) \left. \right| \\ & \leq \left(3 + \frac{2}{\pi} \right) \frac{\sin(\pi s_a)}{M}. \quad (1.10) \end{aligned}$$

Consider now the left hand side of (1.10). Remembering that $M \geq 3$, and since $2x/\pi \leq \sin x \leq x$ for $x \in [0, \frac{\pi}{2}]$, we get

$$\left| \ln \left(\sin^{-1} \left(\frac{\pi(1+s_\alpha)}{M} \right) \sin^{-1} \left(\frac{\pi(1-s_\alpha)}{M} \right) \right) - 2 \ln M \right| \leq \ln(\pi^2). \quad (1.11)$$

Thus by (1.10) and (1.11) we get the final estimate (1.9). \square

1.3.2 Worst-Average Error

From Corollary 1 and Theorem 2 we get sharp estimates on the worst-average error of the **QS** algorithm.

Theorem 3. *Let $M \geq 3$. Then the worst-average error of the **QS** algorithm satisfies the following bounds.*

- For $q \in (1, \infty)$,

$$e_q^{\text{wor-avg}}(M) \leq \frac{1}{M^{1/q}} \left(\frac{1}{\pi} \int_0^\pi \sin^{q-2}(x) dx \right)^{1/q} (1 + o(1)). \quad (1.12)$$

The last estimate is sharp, i.e.,

$$e_q^{\text{wor-avg}}(M) = \Theta \left(\frac{1}{M^{1/q}} \right). \quad (1.13)$$

In particular, for $M - 2$ divisible by 4 we have

$$e_q^{\text{wor-avg}}(M) \geq \frac{1}{M^{1/q}} \left(\frac{1}{\pi} \int_0^\pi \sin^{q-2}(x) |\cos(x)|^q dx \right)^{1/q} (1 + o(1)), \quad (1.14)$$

and the ratio of the integrals in (1.12) and (1.14) are approximately 1 for q close to 1.

- For $q = 1$,

$$e_1^{\text{wor-avg}}(M) \leq \frac{2}{\pi} \frac{\ln M}{M} + \frac{3\pi + 2 + \ln(\pi^2)}{M\pi}. \quad (1.15)$$

This estimate is sharp, i.e.,

$$e_1^{\text{wor-avg}}(M) = \Theta(M^{-1} \ln M). \quad (1.16)$$

In particular, for $M - 2$ divisible by 4 we have

$$e_1^{\text{wor-avg}}(M) \geq \frac{2}{\pi} \frac{\ln M}{M} - \frac{3\pi + 2 + \ln(\pi^2)}{M\pi}.$$

Proof. Consider first the case $q \in (1, \infty)$. By Corollary 1 we have for all $f \in \mathbb{B}_N$,

$$e_q^{\text{avg}}(f, M) \leq \frac{1}{M^{1/q}} \left(\frac{1}{\pi} \int_0^\pi \sin^{q-2}(x) dx \right)^{1/q} (1 + o(1)),$$

where $o(1)$ is independent of f . This yields (1.12).

The estimate (1.12) is sharp since we can take a Boolean function f such that $s_{a_f} \approx \frac{1}{2}$. Then (1.6) yields (1.13). In particular, for $M = 4k + 2$ and $a_f = 1/2$ we have $\theta_{a_f} = \pi/4$, $\sigma_{a_f} = M/4 = k + 1/2$ and $s_a = \frac{1}{2}$. Therefore

$$e_q^{\text{avg}}(f, M) = \frac{1}{M^{1/q}} \left(\frac{1}{\pi} \int_0^\pi \sin^{q-2}(x) |\cos(x)|^q dx \right)^{1/q} (1 + o(1))$$

which proves (1.14). For q close to 1, the value of $\int_0^\pi \sin^{q-2}(x) dx$ is mostly due to the integrand values close to 0 and π . Since $|\cos(x)|^q$ is then approximately equal to one, the ratio of the upper and lower bound integrals is about 1.

For $q = 1$ the estimate (1.15) follows directly from Theorem 2. To prove (1.16) it is enough to choose a Boolean f for which the numbers

$$\sin^2(\pi s_{a_f}) \sin(2\theta_{a_f}) = \sin^2(M\theta_{a_f}) \sin(2\theta_{a_f})$$

are uniformly (in M) separated from 0, see Theorem 2. More precisely, since a_f can take any value k/N for $k = 0, 1, \dots, N$, we take a Boolean function f such that $|a_f - \sin^2(\pi/4 + \pi/(5M))| \leq 1/(2N)$. For sufficiently large N , we have $\theta_{a_f} \approx \frac{1}{4}\pi + \frac{1}{5M}\pi$. For large $M = 4k + \beta$ with $\beta \in \{0, 1, 2, 3\}$, we then have

$$\sin^2(M\theta_{a_f}) \sin(2\theta_{a_f}) \approx \sin^2\left(\frac{4+5\beta}{20}\pi\right) \sin\left(\frac{1}{2}\pi + \frac{1}{2.5M}\pi\right) > c > 0,$$

for some c independent of M .

In particular, for $M - 2$ divisible by 4 we take $N > M$ and a Boolean function $f \in \mathbb{B}_N$ with $a_f = 1/2$. Then

$$s_a = \frac{1}{2} \quad \text{and} \quad \sin^2(\pi s_a) \sin(2\theta_a) = 1,$$

which leads the last estimate of Theorem 3. \square

1.3.3 Quantum Summation Algorithm with Repetitions

The success probability of the **QS** algorithm is increased by repeating it several times and taking the median of the outputs as the final output, see e.g., [2]. We show in this section that this procedure also leads to an improvement of the worst-average error estimate.

We perform $2n + 1$ repetitions of the **QS** algorithm for some $n \in \{0, 1, \dots\}$. We obtain $\sin^2(\pi j_1/M), \sin^2(\pi j_2/M), \dots, \sin^2(\pi j_{2n+1}/M)$ and let $\bar{a}_{n,f}$ be the

median of the obtained outputs, i.e., the $(n + 1)$ st number in the ordered sequence. Let $\rho_{n,f}(\alpha)$, $\alpha \in \mathcal{A}_M$, be the probability that the median $\bar{a}_{n,f}$ is equal to α . This probability depends on the distribution function F_f of the original outputs from $\mathcal{A}_M = \{\sin^2(\pi j/M) : j = 0, 1, \dots, M - 1\}$, which is defined as

$$F_f(\alpha) = \begin{cases} \sum_{\alpha' \in \mathcal{A}_M, \alpha' < \alpha} \rho_f(\alpha') & \text{for } \alpha > 0, \\ 0 & \text{for } \alpha = 0. \end{cases}$$

It is known, see [8] p. 410, that the distribution of the median $\bar{a}_{n,f}$ is of the form

$$\rho_{n,f}(\alpha) = (2n + 1) \binom{2n}{n} \int_{F_f(\alpha)}^{F_f(\alpha) + \rho_f(\alpha)} t^n (1 - t)^n dt, \quad \forall \alpha \in \mathcal{A}_M. \quad (1.17)$$

We are now ready to estimate the worst-average error of the **QS** algorithm with $2n + 1$ repetitions

$$e_{q,n}^{\text{wor-avg}}(M) = \sup_{N > M} \max_{f \in \mathbb{B}_N} \left(\sum_{\alpha \in \mathcal{A}_M} \rho_{n,f}(\alpha) |a_f - \alpha|^q \right)^{1/q}, \quad q \in [1, \infty).$$

We estimate $e_{q,n}^{\text{wor-avg}}(M)$ by using Theorem 12 of [1] which states that the **QS** algorithm with M queries computes \bar{a}_f such that

$$|a_f - \bar{a}_f| \geq c_1 \frac{k}{M} \quad \text{with probability at most } \frac{c_2}{k}$$

for any positive integer k , Here c_1 and c_2 are absolute constants and f is any Boolean function from \mathbb{B}_N . If

$$|a_f - \bar{a}_{n,f}| \geq c_1 \frac{k}{M},$$

then for at least n outcomes $\bar{a}_f(j_1), \dots, \bar{a}_f(j_n)$ we must have

$$|a_f - \bar{a}_f(j_l)| \geq c_1 \frac{k}{M} \quad \text{for } l = 1, \dots, n.$$

But the probability that this occurs is bounded by

$$\binom{2n + 1}{n} \left(\frac{c_2}{k} \right)^n.$$

It follows then that (with c which may depend on n)

$$\text{Prob} \{ |a_f - \bar{a}_{n,f}| \geq c_1 k/M \} \leq c k^{-n}.$$

We now use the standard summation by parts. Define

$$p_k = \text{Prob} \{ c_1(k-1)/M \leq |a_f - \bar{a}_{n,f}| < c_1 k/M \}.$$

Then, by the above estimate we get for an arbitrary integer l ,

$$\sum_{k>l} p_k \leq c l^{-n}.$$

Therefore

$$\begin{aligned} e_{q,n}^{\text{wor-avg}}(M)^q &\leq \sum_{k=1}^{\infty} p_k (c_1 k/M)^q = (c_1/M)^q \sum_{k=1}^{\infty} p_k \sum_{l=1}^k (l^q - (l-1)^q) \\ &= (c_1/M)^q \sum_{l=1}^{\infty} (l^q - (l-1)^q) \sum_{k=l}^{\infty} p_k \\ &\leq c M^{-q} \sum_{l=1}^{\infty} l^{q-1-n} \leq c M^{-q} \end{aligned}$$

for $n > q$ and with the number $c = c_{q,n}$ depending only on q and n . In fact, taking $n = \lceil q \rceil + 1$ it is easy to check that $c_{q,n}$ is a single exponential function of q . Hence, by taking the q th root we have

$$e_{q,n}^{\text{wor-avg}}(M) \leq c_{q,n}^{1/q} M^{-1}$$

with $c_{q,n}^{1/q}$ of order 1. Therefore we have proven the following theorem.

Theorem 4. *The worst-average error of the median of $2(\lceil q \rceil + 1) + 1$ repetitions of the **QS** algorithm with M quantum queries satisfies*

$$e_{q,\lceil q \rceil + 1}^{\text{wor-avg}}(M) = O(M^{-1})$$

with an absolute constant in the big O notation independent of q and M . \square

The essence of Theorem 4 is that the number of repetitions of the **QS** algorithm is *independent* of M and depends only linearly on q . Still, it allows to essentially improve the worst-average error of the **QS** algorithm. As we already mentioned in the introduction, the bound of order M^{-1} is a lower bound on the worst-average error of any quantum algorithm. Hence, the **QS** algorithm with repetitions enjoys optimality also in the worst-average setting.

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